Dempster, Shafer, and Aggregate Uncertainty

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Outline

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Dempster

Evidence

- We are given the probabilities of some collection of subsets of a finite reference set X
- We want to determine the underlying probability distribution $\langle p_x \rangle$ that gave rise to the evidence.
- In general, the problem is underdetermined

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Dempster

Upper and Lower Probabilities

- The lower probability (LP) of a set is the minimum probability of A that is consistent with the evidence.
- The upper probability (UP) of A is the maximum probability for A that is consistent with the data.

Definition (consistent)

A probability distribution $\langle p_x \rangle$ on a finite X is (Dempster) consistent with the evidence if, for every subset A of X we have that

$$LP(A) \leq P(A) \leq UP(A) . \quad (1)$$

Shafer

Evidence Theory

- Shafer's development of Evidence Theory reinterprets Dempster.
- The lower probability is called the Belief measure (Bel)
- The upper probability is called the Plausibility (Pl)
- Shafer defines a function *m*, called a *basic probability assignment* or **bpa**.
- $m : \mathscr{P}(X) \to [0,1]$ is a function which is required to satisfy two conditions: $m(\emptyset) = 0$ and $\sum_{A \in \mathscr{P}(X)} m(A) = 1$.

Shafer

basic probability assignment

- The Belief is given by the formula

$$\text{Bel}(A) = \sum_{B \subseteq A} m(B)$$

- The Plausibility by

$$\text{Pl}(A) = \sum_{B \cap A \neq \emptyset} m(B) \; .$$

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Shafer

Definition (focal)

A *focal element* is a set $A \in \mathscr{P}(X)$ for which $m(A) \neq 0$ and $\mathscr{F}$ denotes the set of all focal elements.

Definition (body of evidence)

A *body of evidence* is denoted
$\mathscr{B} = \langle \mathscr{F}, m \rangle$.

Definition (consistent probability space)

Given a body of evidence $\mathscr{B} = \langle \mathscr{F}, m \rangle$ we define $\mathbf{P}_{\mathscr{B}}$ to be the set of all probability distributions on X that are consistent with $\mathscr{B}$ (see Eq. (1)).

inf/sup strategy

Back and forth

- Many *formulas* in Shafer's Theory of Evidence can be derived by applying a simple back and forth strategy.
- Associate Beliefs/Plausibilities with Lower and Upper Probabilities
- Find the inf / sup of the *probabilistic formula* over the space of all consistent probabilities, $\mathbf{P}_{\mathscr{B}}$ (Dempsterize).
- Translate the inf / sup forward into Evidence theory to get the Belief/Plausibility formula that corresponds to the *probabilistic formula* (Shaferize).

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Shannon Entropy

Entropy⇒AU

- If we apply the inf / sup strategy to the Shannon entropy[**?**]:

$$H(p) = -\sum_{x \in X} p(x) \log_2 p(x) \tag{2}$$

- We produce the Aggregate Uncertainty measures (AU)

$$\begin{aligned} AU(\mathscr{B}) &= \sup_{\mathbf{P}_{\mathscr{B}}} \left[-\sum_{x \in X} p(x) \log_2 p(x) \right] \\ &= \sup_{p \in \mathbf{P}_{\mathscr{B}}} H(p) \end{aligned} \tag{3}$$

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Equivalence

Evidential Equivalence

- Let $\mathscr{F}$ be the focal set of of a body of evidence $\mathscr{B} = \langle \mathscr{F}, m \rangle$.
- Define the lower equivalence relation

$$a \approx b \text{ iff } \forall A \in \mathscr{F}\ a \in A \leftrightarrow b \in A \tag{4}$$

- a is equivalent to b iff every focal set that contains a contains b and vice versa.
- $\mathscr{R}_{\mathscr{F}}$ is the partition induced by $\approx$, and

$$\mathscr{R}_{\mathscr{F}} = \{[a]_{\mathscr{F}} \mid a \in X\}. \tag{5}$$

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Exact

Focal set A

- Every focal set $A \in \mathscr{F}$ is the union of elements of $\mathscr{R}_{\mathscr{F}}$
- In terms of rough set theory, we say that $\mathscr{F}$ is $\mathscr{R}_{\mathscr{F}}$–exact.
- In addition, $\mathscr{R}_{\mathscr{F}}$ is the *roughest* such partition.

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Pignistic Distribution

Pignistic = Betting

- Divide the evidence equally across focal sets
- For a singleton set $A = \{x\}$, we have

$$\mathrm{BetP}(\{x\}) = \sum_{B|x \in B} \frac{m(B)}{|B|} . \tag{6}$$

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Outline

Algorithm

AU Algorithm for Calculating AU from a body of evidence $\mathscr{B} = \langle \mathscr{F}, m \rangle$.

Step 1 Find a non-empty set $A \subseteq X$, such that $\frac{\mathrm{Pl}(A)}{|A|}$ is minimal.

Step 2 Remove any B from $\mathscr{F}$ if $A \cap B \neq \varnothing$ and add A to create $\mathscr{F}_A$.

Step 3 Create $\mathscr{B}_A = \langle \mathscr{F}_A, m_A \rangle$ where $m_A(A) = \mathrm{Pl}(A)$ and $m_A(B) = m(B)$ for all B in $\mathscr{F}_A - \{A\}$.

Step 4 Put $X = X - A$ and $\mathscr{B} = \mathscr{B}_A$.

Step 5 If $X \neq \emptyset$ go to Step 1.

Step 6 $\mathscr{B}_{AU} = \mathscr{B}$.

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Advantages

Each step

- eliminates anything that intersects with A
- produces a higher probability than the previous step
- generates a standard body of evidence

The resulting distribution **p** is pignistic and harmonious.

Outline

The proof

Outline of the proof

Assume there is a consistent distribution **q** with higher entropy

- Every consistent distribution can be produced by arbitrarily dividing $m(A)$ over A
- From this fact we infer that we can go from **p** to **q** in a series of steps where we add α to probability p_x and subtract it from another p_y
- To increase the entropy $p_x < p_y$
- To maintain consistency $p_x > p_y$

Outline

Oops

Example (Demspter)

- Let $X = \{a, b, c, d\}$ with *evidence* $P(\{a, b\}) = P(\{b, c\}) = \frac{2}{3}$.
- The distributions must be of the form $\langle \frac{2}{3} - \beta, \beta, \frac{2}{3} - \beta, \beta - \frac{1}{3} \rangle$ where $\beta \in \left[\frac{1}{3}, \frac{2}{3}\right]$.
- Calculated lower and upper probabilities and consider these as Bel and Pl.
- The algorithm gives $p_{AU} = \langle {}^{1}/{3}, {}^{1}/{3}, {}^{1}/{6}, {}^{1}/{6} \rangle$ with $AU = 1.9183$.
- Unfortunately, the distribution is inconsistent with the evidence.
- And it does not maximize the Shannon entropy, $\beta = {}^{4}/{9}$ does.

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Dempster $\neq$ Shafer

Caution

- Shafer is a subset of Dempster
- The *inf/sup method* assumes an equivalence
- In Dempster we can not arbitrarily distribute the **bpa** over the focal sets