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Fuzzy Geometry: Applied to Comparative Politics

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Abstract. In this paper, we discuss and extend the fuzzy geometries developed by Rosefeld, Buckley, and Eslami. Our purpose is to propose a fuzzy geometry that can be applied to the study of spatial models in comparative politics.

1 Introduction

Comparative politics has been making increasing use of spatial models to derive and test hypotheses on political outcomes resulting from institutional design in democratic systems of governance. These models have been employed in several research agendas to include cabinet formation, durability, and dissolution; policy stability; and party entry. Spatial models typically use crisp logic to locate the exact, ideal policy preferences of political players and institutions. Moreover they assume Euclidean distance in order to simplify estimates of equilibria. Largely as a consequence of the precision imposed by these assumptions, cycling is endemic to spatial models. That is, any point in space can be majority defeated by at least one other point. In the absence of a majority-rule equilibrium, the models can not make a prediction. In order to deal with the problem, spatial modelers have resorted to calculating the uncovered set or imposing further assumptions on the models. The former is non-intuitive and difficult to calculate while the latter often move the models a considerable distance from political reality.

We believe that fuzzy set theory offers a plausible alternative to dealing with cycling. Like crisp logic, a fuzzy approach imposes a set of assumptions on the models that may or may not be appropriate in all cases. However, these assumptions seem particularly appropriate for modeling the preferences of individuals or aggregates of individuals in institutional settings. Indeed, fuzzy logic was explicitly developed to deal with the type of ambiguity and vagueness in human thinking that is associated with such preferences.

Applying fuzzy set theory to spatial models in comparative politics will require a fuzzy geometry. We set ourselves to the task of developing such a geometry in this paper.

Much of the early work in fuzzy geometry was by A. Rosenfeld, [9], [10], and [11]. His writings in fuzzy geometry had broad applications in pattern recognition and image understanding. A discussion of his work in fuzzy geometry can be found in [8]. We discuss briefly only Rosenfeld's work pertaining to fuzzy rectangles and fuzzy triangles since at this moment of development, these results seem more applicable to political science than his other results.

In [1] and [2], Buckley and Eslami developed a fuzzy plane geometry, where fuzzy points were represented by right circular cones. In [3] and [4], the fuzzy geometry in [1] and [2] was modified to use fuzzy points which were represented by frustrums and pyramids. These types of fuzzy points were applied in [3] and [4] to problems involving comparative politics. In this paper, we develop a theory which unifies the various types of fuzzy points so that a general fuzzy plane geometry can be developed. The work of Buckley and Eslami have been found to be quite useful in applying fuzzy geometry to political science.

A fuzzy subset $\widetilde{A}$ of a set X is a function of X into the closed interval $[0,1]$. The support of $\widetilde{A}$, written $\mathrm{Supp}(\widetilde{A})$, is defined to be the set $\{x \in X \mid \widetilde{A}(x) > 0\}$. For $t \in [0,1]$, a t-cut (or t-level set) of $\widetilde{A}$, written $\widetilde{A}^t$, is the set $\{x \in X \mid \widetilde{A}(x) \geq t\}$. We let $\wedge$ denote minimum (or infimum) and $\vee$ denote maximum (or supremum).

2 Fuzzy Rectangles and Fuzzy Triangles

In this section, we briefly discuss the work of Rosenfeld concerning fuzzy rectangles and fuzzy triangles. This will lay the foundation for a fuzzy geometry that we can use to depict policy preferences in spatial models.

A fuzzy subset $\widetilde{A}$ of the plane is called

1. fuzzy connected if for all points P and Q, there is an arc ρ from P to Q such that $\widetilde{A}(R) \geq \widetilde{A}(P) \wedge \widetilde{A}(Q)$;
2. fuzzy convex if for all points P and Q and all points R on the line segment $\overline{PQ}$, $\widetilde{A}(R) \geq \widetilde{A}(P) \wedge \widetilde{A}(Q)$; and
3. fuzzy orthoconvex if for all points P and Q such that the line segment $\overline{PQ}$ is horizontal or vertical and for all points R on $\overline{PQ}$, $\widetilde{A}(R) \geq \widetilde{A}(P) \wedge \widetilde{A}(Q)$.

Let S denote a set. A fuzzy subset $\widetilde{A}$ of S is called separable if there exists a coordinate system (x,y) on S and two fuzzy subsets $\widetilde{B}$ and $\widetilde{C}$ of the x-axis and the y-axis, respectively, such that for all x,y, $\widetilde{A}(x,y) = \widetilde{B}(x) \wedge \widetilde{C}(y)$.

Theorem 2.1. Let $\widetilde{A}$ be a separable fuzzy subset of S. Then the following properties are equivalent.

1. $\widetilde{A}$ is fuzzy connected.
2. $\widetilde{A}$ is fuzzy convex.
3. $\widetilde{A}$ is fuzzy orthoconvex.

Definition 2.1. Let $\widetilde{A}$ be a separable fuzzy subset of S. Then $\widetilde{A}$ is called a fuzzy rectangle if $\widetilde{A}$ satisfies (1)-(3) of Theorem 2.1.

Theorem 2.2. Let $\widetilde{A}$ be a fuzzy subset of S. Then $\widetilde{A}$ is a fuzzy rectangle if and only if there exists a coordinate system and convex subsets $\widetilde{B}$ and $\widetilde{C}$ of the x-axis and y-axis, respectively, such that for all $x,y,\widetilde{A}(x,y) = \widetilde{B}(x) \wedge \widetilde{C}(y)$.

Theorem 2.3. Let $\widetilde{A}$ be a fuzzy subset of S. Then $\widetilde{A}$ is a fuzzy rectangle if and only if $\widetilde{A}^t$ is a rectangle (possibly degenerate or infinite in one or both directions) for all $t \in [0,1]$.

Let θ be a direction in S. Let (x_θ, y_θ) be Cartesian coordinates with x_θ measured along θ and y_θ measured perpendicular to θ. Let $\widetilde{A}$ be a fuzzy subset of S. Then $\widetilde{A}$ is called a fuzzy halfplane if there exists a direction θ and a fuzzy subset $\widetilde{B}$ of the line such that (1) $\widetilde{A}(x_\theta, y_\theta) = \widetilde{B}(x_\theta)$ for all x_θ, y_θ and (2) $\widetilde{B}$ is monotonically nonincreasing. It follows that $\widetilde{A}$ is a fuzzy halfplane if and only if $\{\widetilde{A}^t \mid t \in [0,1]\}$ is a set of nested halfplanes. It also follows that a fuzzy halfplane is fuzzy convex.

Let $\widetilde{A}_1, \ldots, \widetilde{A}_k$ be fuzzy halfplanes whose associated directions $x_1, \ldots, x_k$ are in cyclic order (modulo 2π). If every pair of successive directions (modulo k) differs by less than π, we call $\widetilde{A}_1 \cap \ldots \cap \widetilde{A}_k$ a fuzzy convex polygon. Since an infimum of fuzzy convex sets is fuzzy convex, it follows that a fuzzy convex polygon is a convex fuzzy subset. Consequently, $\widetilde{A}_1 \cap \ldots \cap \widetilde{A}_k$ is a fuzzy convex polygon if and only if $\{\widetilde{A}_1 \cap \ldots \cap \widetilde{A}_k)^t \mid t \in [0,1]\}$ is a set of nested polygons.

Proposition 2.1. A fuzzy subset $\widetilde{A}$ of the line is fuzzy convex if and only if there exists fuzzy subsets $\widetilde{B}$ and $\widetilde{C}$ of the line such that $\widetilde{A} = \widetilde{B} \cap \widetilde{C}$, where $\widetilde{B}$ is monotonically nonincreasing and $\widetilde{C}$ is monotonically nondecreasing.

Theorem 2.4. A fuzzy rectangle is a fuzzy convex polygon.

We consider fuzzy triangles next. Here a fuzzy half plane is defined as before except that we assume $\widetilde{A}$ is montonically nondecreasing. One could have just as easily used this definition in the development of fuzzy rectangles. Once again a fuzzy halfplane is fuzzy convex

Let a, β, γ be three directions in S which are not all contained in a halfplane. Let $\widetilde{A}, \widetilde{B}, \widetilde{C}$ be fuzzy halfplanes in directions α, β, γ, respectively. To avoid degenerate cases, it is assumed that $\widetilde{A}, \widetilde{B}, \widetilde{C}$ are all nonconstant and all take on the value 0. Then $\widetilde{A} \cap \widetilde{B} \cap \widetilde{C}$ is called a fuzzy triangle.

Proposition 2.2. A nonempty level set of a fuzzy triangle $\widetilde{A} \cap \widetilde{B} \cap \widetilde{C}$ is a triangle with sides parallel to α, β, γ.

Let $\widetilde{A}, \widetilde{B}, \widetilde{C}$ be finite valued. Suppose that the fuzzy triangle $\widetilde{T} = \widetilde{A} \cap \widetilde{B} \cap \widetilde{C}$ takes on the values $0 < t_1 < \ldots < t_n \leq 1$. Then $\widetilde{T}$ can be specified by defining a nest of triangles $T_i, i = 1, \ldots, n$, each of which has sides its perpendicular to α, β, γ. On the innermost nonempty triangle $T_n, \widetilde{T}$ has value t_n; on the remaining part of the triangle $T_{n-1}, \widetilde{T}$ takes on the value $t_{n-1}; \ldots$; on the remaining outermost triangle $T_1, \widetilde{T}$ on the value t_1; on the remaining part of the plane $\widetilde{T}$ takes on the value 0. The T_i can be irregularly placed, as long as they are parallel-sided and nested. Thus the T_i are similar.

The sup projections of a fuzzy subset $\widetilde{A}$ onto a line L is a fuzzy subset of L whose value at $p \in L$ is the sup of the values of $\widetilde{A}$ on the line perpendicular to L at p. Thus the projection of $\widetilde{T}$ onto the line L_α perpendicular to α is a "wedding cake" function whose outermost (nonzero) layer has height t_1 and length equal to the side of T_1 perpendicular to α; the successive inner layers have lengths and positions along L_α equal to the lengths and positions (in the direction along L_α) of the corresponding sides of the successive $T_i, i = 2, \ldots, n$. Since sides of the T_is may coincide, some of the step "widths" of the wedding cake may be 0, i.e., some of the step heights may be differences between nonconsecutive t_is.

3 Fuzzy Numbers

The shapes of triangles and rectangles are relatively precise. Their precision may often exceed the ability of individuals or collections of individuals in political situations to precisely define their preferences in two-dimensional policy space. The concept of a fuzzy number helps us to move away from such overly precise representations. There are several equivalent ways to define a fuzzy number. We use the one in [7].

Definition 3.1. Let $\widetilde{N}$ be a fuzzy subset of $\mathbb{R}$. Then $\widetilde{N}$ is called a (real) fuzzy number if the following conditions hold:

i. $\widetilde{N}(x) = 1$ for some $x \in \mathbb{R}$;
ii. $\widetilde{N}^t$ is a closed bounded interval for all $t \in (0,1]$;
iii. Supp$(\widetilde{N})$ is bounded.

This definition is equivalent to the following one.

Definition 3.2. Let $\widetilde{N}$ be a fuzzy subset of $\mathbb{R}$. The $\widetilde{N}$ is called a (real) fuzzy number if the following conditions hold:

i. $\widetilde{N}$ is upper semi-continuous;
ii. there exists $c,d \in \mathbb{R}$ with $c \leq d$ such that $\forall x \notin [c,d], \widetilde{N}(x) = 0$;
iii. there exists $a,b \in \mathbb{R}$ such that $c \leq a \leq b \leq d$ and $\widetilde{N}$ is increasing on $[c,a]$ and $\widetilde{N}$ is decreasing on $[b,d]$, and $\widetilde{N}(x) = 1 \forall x \in [a,b]$.

It follows that $\forall t \in [0,1]$ that if $\widetilde{N}$ is a fuzzy number, then $\widetilde{N}^t$ is a bounded closed interval.

Definition 3.3. Let $(u,v) \in \mathbb{R}^2$ and let $\widetilde{P}$ be a fuzzy subset of $\mathbb{R}^2$. Then $\widetilde{P}$ is called a fuzzy point at (u,v) with respect to a compact and convex set $\mathscr{C}$ containing (u,v) if the following conditions hold:

i. $\widetilde{P}$ is upper semi-continuous;
ii. for all $(x,y) \in \mathbb{R}^2, \widetilde{P}(x,y) = 1$ if and only if $(x,y) \in \mathscr{C}$;
iii. $\forall t \in [0,1], \widetilde{P}^t$ is a compact, convex subset of $\mathbb{R}^2$;
iv. Supp$(\widetilde{P})$ is bounded.

This definition can be generalized to any number of dimensions.

Theorem 3.1 (Klir, Yuan). Let $\widetilde{N}$ be a fuzzy subset of $\mathbb{R}$. Then $\widetilde{N}$ is a fuzzy number if and only if there exists $c,a,b,d \in \mathbb{R}$ such that $c \leq a \leq b \leq d$ and there exists a function $l : (-\infty,a) \to [0,1]$ that is monotonic increasing, continuous from the right, $l(x) = 0$ for all $x \in (-\infty,c)$ and a function $r : (b,\infty) \to [0,1]$ that is monotonic decreasing, continuous from the left, $r(x) = 0$ for all $x \in (d,\infty)$, and

$$\widetilde{N}(x) = \begin{cases} 1 \text{ if } x \in [a,b], \\ l(x) \text{ if } x \in (-\infty,a), \\ r(x) \text{ if } x \in (b,\infty). \end{cases}$$

4 Fuzzy Geometry

In the remainder of the paper, we develop a theory which unifies the various types of fuzzy points so that a general fuzzy plane geometry can be developed. Fuzzy points map the regions within which our individual's or collective player's policy preferences are depicted. Higher policy preferences are contained within higher alpha-cuts.

In the following result, we use the notation $C(x_1,x_2,r_1,...,r_k)$ to denote an equation, where x_1,x_2 are real variables and $r_1,...,r_k$ are parameters. Let $\widetilde{C}$ denote a fuzzy number. For all $t \in [0,1]$, let $\Omega(t) = \{(x_1,x_2) \mid C_1(x_1,x_2,r_{11},...,r_{1k_1}),...,C_n(x_1,x_2,r_{n1},...,r_{nk_n}), r_{ij} \in \widetilde{C}^t, i = 1,...,n; j = k_1,...,k_n\}$. We assume that the $C_1(x_1,x_2,r_{11},...,r_{1k_1}),...,C_n(x_1,x_2,r_{n1},...,r_{nk_n})$ are such that $\Omega(t)$ is a compact and convex set containing $(a,b)\forall t \in (0,1]$. Define the fuzzy point at (a,b) by $\forall (x_1,x_2) \in \mathbb{R}^2, \widetilde{P}(x_1,x_2) = \vee\{t \in [0,1] \mid (x_1,x_2) \in \Omega(t)\}$.

Theorem 4.1. $\forall t \in [0,1], \widetilde{P}^t = \Omega(t)$.

Proof. We have that $(x_1,x_2) \in \widetilde{P}^t$ if and only if there is a sequence $\{t_i\}_{i=1}^{\infty}$ in $[0,1)$ such that $\{t_i\}_{i=1}^{\infty}$ converges to t and $(x_1,x_2) \in \Omega(t_i)$ for each $i = 1,2,...$. (By passing to a subsequence, if necessary, we will assume that $t_i \downarrow t$ as $i \to \infty$.) Consequently, (x_1,x_2) satisfies the conditions

$$C_1(x_1,x_2,r_{11},...,r_{1k_1}),...,C_n(x_1,x_2,r_{n1},...,r_{nk_n})$$

for $r_{mj} \in \widetilde{C}^{t_i}$ for $m = 1,...,n$ and $j = k_1,...,k_n$. Since $\widetilde{C}$ is a fuzzy number, there exist functions l and r satisfying the conditions in Theorem 3.1. Suppose $l(t)$ exists and let x_t denote $l^{-1}(t)$. We know that l is right continuous at x_t and so for some $\delta > 0, l$ is continuous on $[x_t, x_t+\delta)$. For large enough $i \in \mathbb{N}, t_i \in [t, l(x+\delta))$ and so $l^{-1}(t_i) \to l^{-1}(t) = x_t$ as l is continuous here. Essentially the same argument shows that $r^{-1}(t_i) \to r^{-1}(t)$ assuming $r^{-1}(t)$ exists. If $l^{-1}(t)$ does not exist, let $y_t = \inf\{l(w) \mid l(w) > t\}$. Then for a sequence $\{t_j\}_{j=1}^{\infty}$ such that $t_j \downarrow y_t$, we repeat the previous argument. Again, the argument is essentially the same for the function r. To finish the proof, we note that $\widetilde{C}^{t_i} = [l^{-1}(t_i), r^{-1}(t_i)]$ when $l^{-1}(t), r^{-1}(t)$ exist. Hence $l^{-1}(t_i) \leq r_{mj}$ for all m,j. Hence $l^{-}(t) \leq r_{11},...,r_{1k_1},...,r_{n1},...r_{nk_n} \leq r^{-1}(t)$ and so $r_{11},...,r_{1k_1},...,r_{n1},...,r_{nk_n} \in \widetilde{C}^t$. If either or both of $l^{-1}(t), r^{-1}(t)$ does not exist, we modify the argument accordingly using the y_t defined above. The same conclusion follows.

Definition 4.1. Suppose that $\widetilde{N}$ is a fuzzy subset of $\mathbb{R}$ such that $\widetilde{N}(x) = 1$ for all $x \in [a,b]$, is a straight line segment from) $(c,0)$ to $(a,1)$, is a straight line segment from $(b,1)$ to $(d,0)$, and $\widetilde{N}(x) = 0$ for all $x \in (-\infty,c) \cup (d,\infty)$. Then $\widetilde{N}$ is called a trapezoidal fuzzy number. We use the notation $\widetilde{N} = (c/a/b/d)$ for such a fuzzy number. If $a = b$, then $\widetilde{N}$ is called a triangular fuzzy number and we use the notation $\widetilde{N} = (c/a/d)$.

It follows easily that trapezoidal and triangular fuzzy numbers are fuzzy numbers.

A natural way to define a fuzzy point in the plane would be as an ordered pair of real fuzzy numbers. However this definition does not give good results for fuzzy lines. Also pictures of fuzzy points under this definition cannot be constructed. Thus we can replace (ii) in Definition 3.3 with $\mathscr{C} = \{(x,y) \mid (x-a_1)^2 + (y-a_2)^2 \leq r^2\}$ for real numbers a_1, a_2, and r. Thus if $\widetilde{P}$ is a fuzzy point at (a_1,a_2), then we can visualize $\widetilde{P}$ as a surface in $\mathbb{R}^3$ through the graph of the equation $z = \widetilde{P}(x,y)$, $(x,y) \in \mathbb{R}^2$.

This type of fuzzy point that is easy to visualize in $\mathbb{R}^2$. If $r = 0$ in (ii) of Definition 3.3, then we have a fuzzy point which is as right circular cone. For $r \neq 0$, we have a frustrum. These types of fuzzy points have been defined in [3] and applied in [4].

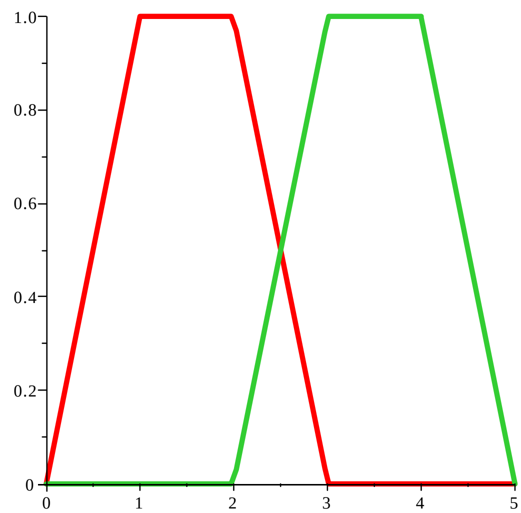

Fig. 1. Trapezoidal Fuzzy Numbers

5 Pyramidal Fuzzy Points

In this section, we consider pyramidal fuzzy points. These points can be used to represent policy preferences in two-dimensional space when political players over each dimension are separable from the other. That is, preferences on each dimension are independent of the other with no trade-off across them. This approach is demonstrated in [4].

A natural way to define a fuzzy point (u, v) in the plane would be as an ordered pair of real fuzzy numbers $(\widetilde{M}, \widetilde{N})$. However this has some problems [1,2]. However we can use $\widetilde{M}$ and $\widetilde{N}$ to construct a fuzzy point P whose membership for an point (x, y) is simply the minimum of $\widetilde{M}(x)$ and $\widetilde{N}(y)$, That is $\widetilde{P}(x, y) = \widetilde{M}(x) \wedge \widetilde{N}(y)$, where $\wedge$ denotes minimum. It is sometimes written $\widetilde{P} = \widetilde{M} \times \widetilde{N}$. The resulting function looks like a truncated pyramid. It is important to note that this construction is reversible so that $\widetilde{P} = \widetilde{M} \times \widetilde{N}$ is separable. If we project $\widetilde{P}$ into the first dimension using the supremum operator, we recover $\widetilde{M}$ and $\widetilde{N}$; $\widetilde{M}(x) = \vee\{\widetilde{P}(x, y) \mid y \in Y\}$ and $\widetilde{N}(y) = \vee\{\widetilde{P}(x, y) \mid x \in X\}$, where $\vee$ denotes supremum.

That this type of fuzzy point models separability can be observed from the fact that the level sets of the fuzzy point are rectangles (and their interiors), the sides of which are parallel to the x and y-axes. Consequently, the sides of the rectangles are defined by equations of the form $x = a$ and $y = b$, which clearly exhibit the independence of x and y. See [4].

Definition 5.1. Let $a_1, a_2 \in \mathbb{R}$ and $\widetilde{C} = (-d/-c/c/d)$ be a trapezoidal fuzzy number, where c, d are nonnegative real numbers. For all $t \in [0, 1]$, let

$$\Omega(t) = \{(x_1, x_2) \mid x_1 - a_1 = r, x_2 - a_2 = s, \text{ where } r, s \in \widetilde{C}^t\}.$$

The fuzzy point $\widetilde{P}$ at (a_1, a_2) is defined as follows:

$$\widetilde{P}(x_1, x_2) = \vee\{t \in [0, 1] \mid (x_1, x_2) \in \Omega(t)\}.$$

We note that for the trapezoidal number $\widetilde{C} = (-d/-c/c/d)$, the line through $(-d,0)$ and $(-c,1)$ is given by $t = (1/(c-d))x - d/(c-d)$ and the line through $(c,1)$ and $(d,0)$ is given by $t = (1/(c-d))x - d/(c-d)$. Solving these equations for x, we get $x = (d-c)t - d$ and $x = (c-d)t + d$, respectively. Thus $\widetilde{C}^t = [(d-c)t - d, (c-d)t + d]$.

The following proposition follows from Theorem 4.1.

Proposition 5.1. For all $t \in [0,1], \widetilde{P}^t = \Omega(t)$.

Now $\widetilde{P}$ defines a pyramid with center at (a_1,a_2) and with height equal to 1. For $t \in [0,1]$,

$$\begin{aligned}\Omega(t) &= \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 - a_1 = r, x_2 - a_2 = s, \text{ where } r,s \in \widetilde{C}^t\} \\ &= \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 - a_1 = r, x_2 - a_2 = s, (d-c)t - d \le r,s \le (-d+c)t + d\} \\ &= [a_1 + (d-c)t - d, a_1 + (-d+c)t + d] \times [a_2 + (d-c)t - d, a_2 + (-d+c)t + d]\end{aligned}$$

Hence $\Omega(0^+) = \{(x_1x_2) \in R^2 \mid x_1 - a_1 = r, x_2 - a_2 = s, -d \le r,s \le d\}$. Thus $\Omega(0^+)$ is not only the set of all points on the square $x_1 - a_1 = -d$, or $x_1 - a_1 = d$, or $x_2 - a_2 = -d$, or $x_2 - a_2 = d$, but also on the interior of the square. Now

$$\begin{aligned}\Omega(1) &= \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 - a_1 = r, x_2 - a_2 = s, \text{ where } r,s \in \widetilde{C}^1\} \\ &= \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 - a_1 = r, x_2 - a_2 = s, \text{ where } r,s \in [-c,c]\}.\end{aligned}$$

We see that we can depict a fuzzy point in two ways. One way is to visualize the fuzzy point in $\mathbb{R}^2$ via $\Omega(t)$. That is, any point in $\Omega(0^+)$ lies on some square concentric to and in the interior of the square, $x_1 - a_1 = -d$, or $x_1 - a_1 = d$, or $x_2 - a_2 = -d$, or $x_2 - a_2 = d$, say on $x_1 - a_1 = r$, or $x_2 - a_2 = r$ for some r such that $-d \le r \le d$. Suppose (b_1,b_2) is on the square. Then $b_1 - a_1 = r$, or $b_2 - a_2 = r$. Hence $b_1 - a_1 = b_2 - a_2$. Thus

$$\widetilde{P}(b_1,b_2) = \begin{cases} 1 \text{ if } b_1 = a_1 + r, b_2 = a_2 + s \text{ for some } r,s \text{ such that } -c \le r,s \le c \\ \qquad \vee\{t \in [0,1] \mid \ (d-c)t - d \le b_1,b_2 \le (c-d)t + d\}. \end{cases}$$

A second way to visualize $\widetilde{P}$ is as a pyramid. Then consider the square $[a_1 + (d-c)t - d, a_1 + (-d+c)t + d] \times [a_2 + (d-c)t - d, a_2 + (-d+c)t + d]$ at a height t on the pyramid that is obtained by intersecting the pyramid with the plane parallel to the (x_1,x_2)-plane at this height, namely t.

Definition 5.2. Let $a_1, a_2 \in \mathbb{R}$ and $\widetilde{C} = (-d,-c,c,d)$ be a trapezoidal fuzzy number, where c,d are nonnegative real numbers. For all $t \in [0,1]$, let

$$\Omega(t) = \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1 - a_1)^2 + (x_2 - a_2)^2 = r^2, r \in \widetilde{C}^t\}.$$

Then a fuzzy point $\widetilde{P}$ at (a_1,a_2) is defined as follows:

$$\widetilde{P}(x_1,x_2) = \vee\{t \in [0,1] \mid (x_1,x_2) \in \Omega(t)\}.$$

We note that for the trapezoidal number $\widetilde{C} = (-d,-c,c,d)$, the line through $(-d,0)$ and $(-c,1)$ is given by $t = (1/(d-c))x + d/(d-c)$ and the line through $(c,1)$ and $(d,0)$ is given by $t = (1/(c-d))x - d/(c-d)$. Solving these equations for x, we get $x = (d-c)t - d$ and $x = (c-d)t + d$, respectively. Thus $\widetilde{C}^t = [(d-c)t - d, (c-d)t + d]$.

The following Proposition follows from Theorem 4.1.

Proposition 5.2. For all $t \in [0,1]$, $\widetilde{P}^t = \Omega(t)$.

Now $\widetilde{P}$ defines a right-circular "cone" with center at (a_1,a_2) and with height equal to 1. For $t \in (0,1]$,

$$\begin{aligned}\Omega(t) &= \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1-a_1)^2+(x_2-a_2)^2 = r^2, r \in \widetilde{C}^t\} \\ &= \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1-a_1)^2+(x_2-a_2)^2 = r^2, (d-c)t-d \le r \le (-d+c)t+d\}.\end{aligned}$$

Hence $\Omega(0^+) = \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1-a_1)^2+(x_2-a_2)^2 = r^2, -d \le r \le d\}$. Thus $\Omega(0^+)$ is not only the set of all points on the circle $(x_1-a_1)^2+(x_2-a_2)^2 = d^2$, but also on the interior of the circle. Now

$$\begin{aligned}\Omega(1) &= \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1-a_1)^2+(x_2-a_2)^2 = r^2, r \in \widetilde{C}^1\} \\ &= \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1-a_1)^2+(x_2-a_2)^2 = r^2, r \in [-c,c]\}.\end{aligned}$$

We see that we can depict a fuzzy point in two ways. One way is to visualize the fuzzy point in $\mathbb{R}^2$ via $\Omega(t)$. That is, any point in $\Omega(0^+)$ lies on some circle concentric to and in the interior of the circle, $(x_1-a_1)^2+(x_2-a_2)^2 = d^2$, say on $(x_1-a_1)^2+(x_2-a_2)^2 = r^2$ for some r such that $-d \le r \le d$. Suppose (b_1,b_2) is on the circle. Then $(b_1-a_1)^2+(b_2-a_2)^2 = r^2$. Hence $r = \pm\sqrt{(b_1-a_1)^2+(b_2-a_2)^2}$. Thus $\widetilde{P}(b_1,b_2) = \begin{cases} 1 \text{ if } (x_1-a_1)^2+(x_2-a_2)^2 \le c^2, \\ (d-\sqrt{(x_1-a_1)^2+(x_2-a_2)^2})/(d-c) \text{ otherwise.}\end{cases}$

A second way to visualize $\widetilde{P}$ is as a right circular "cone." Then consider the circle $(x_1-a_1)^2+(x_2-a_2)^2 = r^2$ at a height $(d-\sqrt{(x_1-a_1)^2+(x_2-a_2)^2})/(d-c)$ on the "cone" that is obtained by intersecting the "cone" with the plane parallel to the (x_1,x_2)-plane and at this height, namely $z = (d-\sqrt{(x_1-a_1)^2+(x_2-a_2)^2})/(d-c)$.

We now need the definition of a fuzzy line segment. Let $\widetilde{P}$ and $\widetilde{Q}$ be distinct fuzzy points. Let $\Omega_l(t)$ (or $\Omega_{\widetilde{P}\widetilde{Q}}(t)$) denote the set of all line segments from a point in $\widetilde{P}^t$ to a point $\widetilde{Q}^t$. The fuzzy line segment $\widetilde{L}_{\widetilde{P}\widetilde{Q}}$ from $\widetilde{P}$ to $\widetilde{Q}$ is defined as follows: $\forall (x_1,x_2) \in \mathbb{R}^2, \widetilde{L}_{\widetilde{P}\widetilde{Q}}(x_1,x_2) = \vee\{t \in [0,1] \mid (x_1,x_2) \in l, l \in \Omega_l(t)\}$. It is known that $(\widetilde{L}_{\widetilde{P}\widetilde{Q}})^t = \Omega_l(t)$ for all $t \in [0,1]$.

Let A,B,C denote ideal points of the players. Then the unanimity core, UC, is the set $\{p \in R^2 \mid d(A,p) > d(A,SQ))$ or $d(B,p) > d(B,SQ)$ or $d(C,p) > d(C,SQ)\}$, where d denotes the usual Euclidean distance function. Hence $SQ \notin UC$. It follows that if A,B,C are not all on the same straight line, then UC is the set of all points on or in the interior of the triangle ABC and excluding the point SQ. Suppose we consider the ideal points as fuzzy points $\widetilde{A},\widetilde{B},\widetilde{C}$, where $\widetilde{A} = (-a/A_1/A_2/a)$, $\widetilde{B} = (-b/B_1/B_2/b)$, $\widetilde{C} = (-c/C_1/C_2/c)$ for $a,b,c > 0$ and where A is the midpoint of the line segment from A_1 to A_2 and similarly for B and C. Then $\widetilde{A},\widetilde{B},\widetilde{C}$ determine a fuzzy triangle $\widetilde{T}$ defined as follows:$\forall (x_1,x_2) \in \mathbb{R}^2$,

$$\widetilde{T}(x_1,x_2) = \widetilde{L}_{\widetilde{A}\widetilde{B}}(x_1,x_2) \vee \widetilde{L}_{\widetilde{B}\widetilde{C}}(x_1,x_2) \vee \widetilde{L}_{\widetilde{C}\widetilde{A}}(x_1,x_2).$$

We define the extended unanimity core EUC to be the set $\cup_{t\in(0,1]}\widetilde{T}^t$. Clearly $EUC = \text{Supp}(\widetilde{T})$. It can be shown that $EUC = \{(x_1,x_2) \in \mathbb{R}^2 \mid \exists t \in (0,1], \exists l \in \Omega_{\widetilde{A}\widetilde{B}}(t) \cup \Omega_{\widetilde{B}\widetilde{C}}(t) \cup \Omega_{\widetilde{C}\widetilde{A}}(t)$ such that $(x_1,x_2) \in l\}$.

In the remainder of the paper, we let D denote the usual distance function in Euclidean space.

Definition 5.3. Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two fuzzy points. $\forall t \in [0,1]$, let $\Omega(t) = \{D(u,v) \mid u \in (\widetilde{P}_1)^t$ and $v \in (\widetilde{P}_2)^t\}$. Define the fuzzy subset $\widetilde{D}(\widetilde{P}_1,\widetilde{P}_2)$ of $\mathbb{R}$ by $\widetilde{D}(\widetilde{P}_1,\widetilde{P}_2)(a) = \vee\{t \in [0,1] \mid a \in \Omega(t)\}\ \forall a \in \mathbb{R}$.

Let $t \in [0,1]$. We note that in Definition 1.7, $\Omega(t)$ is defined in terms of a pair of fuzzy points, say $\widetilde{P}_1, \widetilde{P}_2$. It follows that $\Omega(t) = \{r \in \mathbb{R} \mid \exists u \in (\widetilde{P}_1)^t, \exists v \in (\widetilde{P}_2)^t$ such that $r = D(u,v)\}$.

Theorem 5.1. Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two fuzzy points. Then $\forall t \in [0,1]$, the level set $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)^t = \Omega(t)$. Furthermore, $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)$ is a fuzzy number.

Proof. That $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)^t = \Omega(t)$ for all $t \in (0,1]$ follows in a manner entirely similar to that of the proof of Theorem 4.1 and [1]. We show that $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)$ is a fuzzy number since our definition of fuzzy number is stated differently than that in [1] and since our definition of fuzzy point also differs. For all $t \in (0,1]$, the t-cuts $\widetilde{P}_1^t$and $\widetilde{P}_2^t$ are closed bounded intervals. Thus $\widetilde{P}_1^t$and $\widetilde{P}_2^t$ is closed bounded and convex $\forall t \in (0,1]$. Since D is continuous and compactness is a topological property, it follows that $\Omega(t) = \widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)^t$ is closed and bounded for all $t \in (0,1]$. It also follows that $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)^t$ is convex $\forall t \in (0,1]$. Clearly, $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)(x) = 0$ outside $[c,d]$ and $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)(x) = 1 \forall x \in [a,b]$. Consequently, $\widetilde{D}$ is a fuzzy number.

6 Stair-step Fuzzy Points

To this point, we have assumed a continuous fuzzy number. Continuous fuzzy numbers, however, assume an infinite degree of generality and thereby assume that players can make extremely fine-grained distinctions between policy positions. A more appropriate representation of players' policy preferences is provided by the coarse degree of granularity that can had with discrete fuzzy numbers. We discuss this concept in [12].

In this section, we make use of the following fuzzy number. Let a_i, $i = 1,2,3,4$, be positive real numbers such that $a_1 < a_2 < a_3 < a_4$. Define the fuzzy subset $\widetilde{C}$ of $\mathbb{R}$ as follows: $\forall x \in \mathbb{R}$,

$$\widetilde{C}(x) = \begin{cases} 1 \text{ if } -a_1 \leq x \leq a_1, \\ .75 \text{ if } -a_2 \leq x < -a_1 \text{ or } a_1 < x \leq a_2, \\ .50 \text{ if } -a_3 \leq x < -a_2 \text{ or } a_2 < x \leq a_3, \\ .25 \text{ if } -a_4 \leq x < -a_3 \text{ or } a_3 < x \leq a_4, \\ 0 \text{ otherwise.} \end{cases}$$

Let $\Omega(t) = \{(x_1,x_2) \mid x_1 - a_1 = r, x_2 - a_2 = s$, where $r,s \in \widetilde{C}^t\}$ or $\Omega(t) = \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1 - a_1)^2 + (x_2 - a_2)^2 = r^2, r \in \widetilde{C}^t\}$, where $t \in [0,1]$. In the former case, we get a square wedding cake for a fuzzy point and in the latter case a circular wedding cake for a fuzzy point, where the fuzzy point at (a,b) is defined by $\forall (x_1,x_2) \in \mathbb{R}^2, \widetilde{P}(x_1,x_2) = \vee\{t \in [0,1] \mid (x_1,x_2) \in \Omega(t)\}$.

Proposition 6.1. For all $t \in [0,1]$, $\widetilde{P}^t = \Omega(t)$.

7 Elliptical Fuzzy Points

We now further loosen the restrictions in the shape of fuzzy numbers. In so doing, we permit for modelling of a wider array of policy preferences in comparative politics.

The equation of an ellipse is of the form $(x_1 - h)^2/b^2 + (x_2 - k)^2/a^2 = 1$, where a,b,h,k are real numbers and if $a > b$, the foci are determined by the equation $c^2 = a^2 - b^2$. Thus we use the following approach in defining a fuzzy elliptical fuzzy point. Let $\widetilde{A}, \widetilde{B}, \widetilde{H}, \widetilde{K}$ be fuzzy real numbers such that

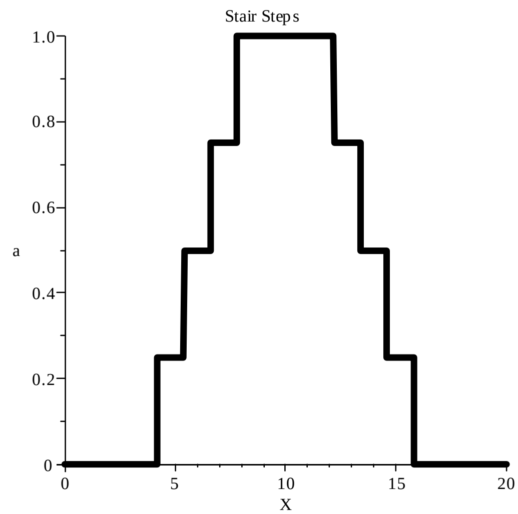

Fig. 2. Stair-step Fuzzy Number

$\forall t \in (0,1], 0 \notin \widetilde{A}^t, 0 \notin \widetilde{B}^t$. For all $t \in [0,1]$, let $\Omega(t) = \{(x_1,x_2) \mid (x_1-h)^2/b^2 + (x_2-k)^2/a^2 = 1, a \in \widetilde{A}^t, b \in \widetilde{B}^t, h \in \widetilde{H}^t, k \in \widetilde{K}^t\}$. Then a fuzzy point $\widetilde{P}_{(h,k)}$ at (h,k) is defined as follows:

$$\widetilde{P}_{(h,k)}(x_1,x_2) = \vee\{t \in [0,1] \mid (x_1,x_2) \in \Omega(t)\}.$$

However it is unnecessary for our model to contain this level of complexity by having the center of the ellipse be defined by fuzzy numbers. Hence we let $\widetilde{H}$ and $\widetilde{K}$ be crisp numbers h and k, respectively. It also suffices for our model that $\widetilde{A}$ and $\widetilde{B}$ be triangular fuzzy numbers. Then $\Omega(t) = \{(x_1,x_2) \mid \frac{(x_1-h)^2}{b^2} + \frac{(x_2-k)^2}{a^2} = 1, a \in \widetilde{A}^t, b \in \widetilde{B}^t\}$ and $\widetilde{P}_{(h,k)}(x_1,x_2) = \vee\{t \in [0,1] \mid (x_1,x_2) \in \Omega(t)\}$ for this latter $\Omega(t)$. The proof of the following theorem is entirely similar to that of Theorem 4.1.

Proposition 7.1. For all $t \in [0,1]$, $\widetilde{P}^t = \Omega(t)$.

Since $\widetilde{A}$ and $\widetilde{B}$ are assumed to be triangular fuzzy numbers, $\widetilde{A}^1 = \{a\}$ and $\widetilde{B}^1 = \{b\}$. Thus $\widetilde{P}^1_{(h,k)} = \{(x_1,x_2) \mid \frac{(x_1-h)^2}{b^2} + \frac{(x_2-k)^2}{a^2} = 1\}$. Thus we see that $\widetilde{P}^1_{(h,k)}$ does not contain the interior of the ellipse. In order to overcome this, we can proceed as follows: $\forall t \in [0,1]$,

$$\Omega(t) = \{(x_1,x_2) \mid \frac{(x_1-h)^2}{b^2} + \frac{(x_2-k)^2}{a^2} \leq 1, a \in \widetilde{A}^t, b \in \widetilde{B}^t\}.$$

Define $\widetilde{P}_{(h,k)}(x_1,x_2) = \vee\{t \in [0,1] \mid (x_1,x_2) \in \Omega(t)\}$. Then as in Theorem 4.1, $\widetilde{P}^t = \Omega(t)$ for all $t \in \Omega(t)$.

Now let $\widetilde{A}$ and $\widetilde{B}$ be trapezoidal fuzzy numbers. Define $\Omega(t)$ and $\widetilde{P}_{(h,k)}$ as we did for triangular fuzzy numbers. To avoid unnecessary complication, we assume $\mathrm{Supp}(\widetilde{A}) \cap \mathrm{Supp}(\widetilde{B}) = \emptyset$. Let $\widetilde{A} = (u/v/w/z)$ and $\widetilde{B} = (i/j/l/m)$, where $m \leq u$. Then

$$\widetilde{P}^1_{(h,k)} = \{(x_1,x_2) \mid \frac{(x_1-h)^2}{b^2} + \frac{(x_2-k)^2}{a^2} = 1, \text{ where } v \le a \le w,\ j \le b \le l\}.$$

It follows that $\widetilde{P}^1_{(h,k)} = \{(x_1,x_2) \mid (x_1-h)^2/b^2 + (x_2-k)^2/a^2 = 1,$where $a^2 = w^2$ and $b^2 = l^2\}$.

Example 7.1. Let $\widetilde{A} = (4/5/6/7)$ and $\widetilde{B} = (1/2/3/4)$. We have that

$$\widetilde{P}^1_{(h,k)} = \{(x_1,x_2) \mid \frac{(x_1-h)^2}{b^2} + \frac{(x_2-k)^2}{a^2} = 1, \text{ where } 5 \le a \le 6,\ 2 \le b \le 3\}.$$

Thus $\widetilde{P}^1_{(h,k)}$ is the ellipse described as follows and part of its interior. The major axis is on the line parallel to the x_2-axis and through the point (h,k) and the minor axis is parallel to the x_1-axis and through the point (h,k). The major and minor vertices are $(h,k\pm 6)$ and $(h\pm 3,k)$, respectively. Since $m = 4 \le 4 = u$, the major axis is always on line a parallel to the x_2-axis and the minor axis is on a line parallel to the x_1-axis. The part of the interior that is not included is the interior of the ellipse, $\frac{(x_1-h)^2}{2^2} + \frac{(x_2-k)^2}{5^2} = 1$.

It is clear in that the above definitions can be modified so that the ellipses have their major axis on a line parallel to the x_1-axis and their minor axis on a line parallel to the x_2-axis.

8 Conclusions

We have introduced several fuzzy geometric concepts that can be used to depict the policy preferences in spatial models in comparative politics. The shapes that we have discussed barely begin to introduce the number of plausible geometric forms that might conceivably be useful for this purpose. The more important question, however, is can the modeling of preferences using fuzzy geometry lead to better predictions in spatial models? To answer this question, once we know the policy preferences of each actor and their relative locations in the policy space, we must model the decision rules and predict the policy outcome. The outcomes must be compared with those from crisp models to see which best stands up to empirical scrutiny.

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Applying Fuzzy Mathematics in Political Science: What Is To Be Done?

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Abstract. We report on our recent work applying fuzzy set theory in political science. In our view, the most promising direction is provided by spatial models. These models have been plagued by chaos. Fuzzy set theory markedly reduces the potential for instability. We conclude with an overview of future issues for research in fuzzy spatial models.

1 Introduction

For most of the last three years, we have been engaged in a very fruitful collaboration. Together we have led a team of faculty and students, both undergraduate and graduate, that has focused on issues related to the application of fuzzy mathematics to political science. The project has resulted thus far in the publication of a book, a major article in political science, and a goodly number of math articles; and several of the student participants have gone on to PhD programs in mathematics and political science at major institutions. While it would be tempting to use the space graciously provided to us by the editors of this newsletter to expound on those successes, we rather suspect that most readers would find it very tiring to read such a self-indulgent essay. Instead, we propose to address what we have learned from our efforts. Our intent is to suggest where those interested in expanding the use of fuzzy mathematics in political science might make the most meaningful contribution.

Reviewing Nurmi and Kacprzyk's [1] concise overview of the research using fuzzy approaches in political science, we are struck by the very small number of articles published in political science journals. Indeed, aside from Charles Ragin [2], there is a dearth of well-known scholars in the discipline who have taken fuzzy mathematics seriously. An exception that proves the rule is Claudio Cioffi-Revilla's [3] lone article arguing for the utility of fuzzy mathematics. When we queried him recently on why he never followed-up on his own argument for fuzzy approaches, he responded that he needed to make tenure at the time and that he subsequently never found the time to return to a serious consideration of fuzzy mathematics. His response is instructive: a fuzzy approach in political science, as in virtually every other discipline, is counter-pardigmatic.

That is one of the reasons why Ragin's fuzzy set theory approach to social science research has never gotten much traction. Ragin offers a fuzzy set approach for hypothesis testing, which permits social scientists to test for necessary and sufficient conditions when there are a limited number of observations. Standard regression analysis requires a large number of cases. Among the few published articles using Ragin's method is Penning's [4] 2003 piece in the European Journal of Political Research. To date, there hasn't been a single application published in the discipline's top four journals: the American Political Science Review, the British Journal of Political Science, the American Journal of Political Science, and the Journal of Politics.

There are many reasons for this. First, the case for the ambiguity and vagueness of the concepts that are being tested is often weak. While democracy, development, presidential strength, and judicial independence, just to name a few, seem fuzzy, they are no more fuzzy than similar concepts in economics. Indeed, the standard approach for testing hypotheses in the social sciences, regression analysis, is able to deal with quite a bit of measurement error, particularly on the dependent variable. Second, as the literature on Ragin's fuzzy method has taken pains to point out [5,6], Ragin's reliance on measures of subsethood assumes that the data are not sensitive to issues of scale. In the absence of a universal dimension, the argument for either necessity or sufficiency, or for neither, may rest solely on measurement error. Third, Bayesian probability approaches not only deal more adequately with measurement error by focusing on the more conventional issue of correlation, they are also able to deal with a small number of observations.

The short of it is that crisp approaches to hypothesis testing in political science are not suffering from a sense of crisis, in the absence of which there is little tolerance for fundamental change. What paradoxes there are, appear to be easily managable within the conventional probability paradigm using crisp data. While we are not arguing against efforts dealing with the property ranking issue that troubles measures of subsethood in order to strengthen Ragin's tests of necessity and sufficiency, we are not at all sanguine that doing so will broaden the appeal of fuzzy approaches to hypothesis testing to any significant extent. Furthermore, it is clear that attempts to evaluate the potential of measures of equality will fall on deaf ears. These approaches fail to deal with necessity and sufficiency and butt up against the strength of statistical analysis, testing for hypothesized correlations in the data.

On the other hand, there is a sense of crisis among those in political science attempting to build formal theories. Formal theoretic approaches in political science rest on the rational actor assumption: political players are utility maximizers. When faced with a set of options, they will rank order the options and choose that alternative closest to their ideal outcome. This permits the spatial modeling of actors' preferences in relation to alternatives. The problem that these spatial models face is that under an assumption of sincere choice (voting), the emergence of an equilibrium (stable prediction) in all but the simplest political situations is a virtual impossibility. The so-called majority cycling problem is encountered by these n–dimensional ($n \geq 2$) crisp models: every alternative is preferred by at least one other alternative by some majority of political actors.

Formal modelers in political science have hithertofore attempted to deal with the problem along two axes. Neither has proven particularly successful as of yet. The first is to model the effect of the political institutions involved in a political process. This approach lies at the base of The New Institutionalism, a major research agenda that attempts to predict political outcomes by assuming that institutions are not neutral. They effect outcomes in significant ways. Modelers attempt to determine the likely set of outcomes based on the preferences of the relevant players and the institutional design within which they are contending over policy choices. Among the foremost scholars in this school is Kenneth Shepsle [7] whose early work with Barry Weingast [8] on the effect of committee systems on legislation in the U.S. Congress was based on the argument that committees are gatekeepers, that is they exercise an effective veto over all legislation.

The second axis is a more thoroughly mathematical one: the uncovered set [9,10,11,12]. If we assume that players behave strategically - that is instead of acting on their preferences at each stage of a selection process, they act in order to maximize the outcome - then the set of outcomes is reduced to a relatively small area of the issue space. The uncovered set, $UC(X)$, is the set of all points that beat every other point in either one or two steps. The alternatives in this region, which together comprise the uncovered set, can be reached by some path. Formally, $x \in UC(X) \Leftrightarrow \forall y \in X \setminus \{x\}$ $(x\ P\ y$ or $\exists z(x\ P\ z$ and $z\ P\ y))$. Unfortunately, the size and shape of the uncovered set is not intuitive; and it is quite difficult to calculate.

Our project suggests a third approach: fuzzy spatial models. Unlike many observable political phenomena, the preferences of human actors, whether as individuals or in institutional settings (e.g., a committee, a legislature, a court), are clearly vague and ambiguous. Hence, a fuzzy approach to spatial modeling would appear to be easily justifiable.

Some very important pioneering work has been published in fuzzy math journals on spatial models. The subject that has drawn the most attention is how to assign membership values to the preferences of collective political institutions. The most common approach is to assign values directly to preference relations [14,15,16,17]. We agree with Nurmi's approach [18]. He aggregates the collective preferences of institutions from those of the individuals operating within them. It is not clear how these collective preferences would be determined apart from individual preferences. Furthermore, formal models in political science are premised on the notion that the individual is the basic unit of analysis. So, we start with the individual.

The question that immediately arises is how to aggregate individual preferences across dimensions. An example would be a legislator faced with voting on a budget that includes funding for both defense and education. What aggregation operator do we use to map her budget preference in two-dimensional space? A minimum aggregation operator (choosing the lowest set inclusion value across the two dimensions) would be appropriate for a legislator who is extremely risk averse, and a maximum aggregation operator (choosing the highest set inclusion value across the two dimensions) would be appropriate for a legislator who is a risk taker. However, both can result in logical absurdities. Say for instance that a risk-prone legislator is faced with a decision. The maximum aggregation operator leads to the conclusion that he would be indifferent in a choice between two options $\{x_i, y_i\}$ whose respective values are $\{0,1\}$ and $\{1,1\}$. It seems implausible that a person who can achieve his maximum preference on both dimensions would be unable to choose that option over an alternative on which he would achieve his maximum preference on one dimension and his worst case scenario on the other. Similarly, suppose that a highly risk-averse legislator is faced with a decision. The minimum aggregation operator results in her being indifferent in a choice between two options whose respective values are $\{.5,1\}$ and $\{.5,.5\}$.

In [13] we argue that an ordered weighted average (OWA) weighted in favor of the smallest value on the two dimensions is the most plausible in most political situations. OWA avoids the absurdities of both the minimal and maximal aggregation operator while capturing what seems to us as obvious: most political actors are relatively risk-averse.

While the argument that preferences are inherently ambiguous is a powerful one, the more compelling argument for political scientists to adopt a fuzzy approach to spatial modeling depends on whether it will yield stable predictions. This is the Achilles' heel of crisp approaches to spatial models. Armed with spatial maps of the preferences of political actors, we can face this challenge.

We are indebted to Nurmi's [19] initial work on this subject. As far as we know, he was the first to attempt to apply a fuzzy approach to spatial models. Nurmi uses the core of a fuzzy number to represent all alternatives that are satisfactory to a political actor. Those outside the core are not satisfactory. He argues that actors whose cores intersect are capable of achieving consensus on an outcome if they satisfice. While this is an important finding, it is achieved by jettisoning the key assumption of spatial models in political science: political actors are utility maximizers (the rational actor assumption), not satisficers.

Nonetheless, Nurmi's argument can be extended to reach some important tentative conclusions. We do so in [13]; and we demonstrate the approach in an empirical analysis of a European cabinet formation process in [20]. In these early works, we discovered that a goodly number of spatial arrangements of preferences that result in cycling in the crisp case do not do so under assumptions of fuzzy preferences, without jettisoning the rational actor assumptions.

In a manuscript under review [22], we provide a formal proof of just how well-behaved fuzzy spatial models are. The ability of spatial models to make predictions about political outcomes rests on the presence of a maximal set. A maximal set is the set of undominated alternatives under majority rule. No alternative in the set is strictly preferred by any other alternative (in, or out, of the set). Formally, $M(R,X) = \{x \in X \mid x \; R \; y \; \forall y \in X\}$ [21, p. 3]. We demonstrate in [22] that in all but a limited number of cases the maximal set is empty if and only if the Pareto set is a union of cycles or a subset of a union of cycles. That is, if the elements in the Pareto set cycle under majority rule or if they constitute a subset of a cycle under majority rule, the maximal set is empty; and vice versa. Furthermore, we are able to completely characterize the exceptions for the three-player game based on a general definition of the exceptions. The exceptions together with the cycles and cycle sets are not likely to occur in spatial models. Hence, the probability of cycling is significantly less in fuzzy spatial models than in their crisp counterparts.

This is a very important finding and one that could potentially open the door to the wider use of fuzzy mathematics by formal modelers in political science. However, before that is likely to occur, many important questions must be resolved. As the Russians have historically been fond of saying,

Что делать?

(chto delat'?) – What is to be done?

While our work thus far indicates that fuzzy spatial models are substantially less prone to cycling, they do not always result in stable predictions. In the absence of a maximal set, can the predictions be reduced to some subset of the policy space by some other method? Crisp models result in a top cycle set when a maximal set does not exist. A top cycle set is defined as:

$$\begin{aligned} T(\rho) = \{x \in X \mid \forall y \in X \setminus \{x\}, \; \exists \{a_0, a_1, \ldots, a_r\} \subseteq X, \\ \text{such that } a_0 = x, \; a_r = y, \; r < \infty \\ \text{and, } \forall t \leq r-1, a_t \; P \; a_{t+1}\}, \end{aligned} \tag{1}$$

where ρ is a set of preference orders, or a preference profile,[3] held by individuals making choices over a set of alternatives [21, p. 169]. However, as McKelvey [23] has observed, under assumptions of sincere voting, the top cycle set can encompass the entire policy space. For this reason, scholars turned to the concept of the uncovered set, which we discussed previously. When sophisticated voting is assumed, then the uncovered set results as the union of the top cycle set and the Pareto

[3] An individual has a preference order. A set of preference orders characterizing a set of individuals is a preference profile.

set. Formally [24 , p. 7], the Pareto set is defined as

$$PS_N(R) = \{x \in X \mid \forall y \in X \ (\exists i \in N, y \ P_{\ i} x \Rightarrow \exists j \in N, x \ P_{\ j} y)\}.$$

What then might be the consequence when fuzzy spatial models do not predict a maximal set? Does a top cycle set emerge? Can the top cycle set encompass the entire set of alternatives? Or does a fuzzy set approach more appropriately model the logic of the uncovered set? It is not at all clear that these same concepts will be applicable to fuzzy models. Our work thus far has demonstrated that fuzzy set theory has a logic of its own. It seldom suffices merely to overlay fuzzy logic on crisp concepts. For example, in [20] we found counter-intuitively that while fuzzy points may take any shape when trade-offs are permitted across the dimensions defining an issue (non-separability), in separable issue space they are always rectangluar.

While these represent the more immediate questions, the more important task is to fuzzify the set theoretical underpinnings of spatial modeling. Not only will the answers to our questions concerning the existence of fuzzy top cycle sets and uncovered sets emerge out of this work, but political scientists will be given the formal tools to develop fuzzy models. We have begun some of this work using as our guide Austen-Smith and Banks' [21,24] authoritative two-volume formal treatment of spatial modeling in political science. Thus far, we have done work on Arrow's Theorem [25] as well as the link between observable choices made by political players and their preferences (whether a choice function is rationalizable) [26]. There remains yet much to do.

Lotfi Zadeh comments in his foreword to our book on spatial modeling [13] that he had thought that given the inherent ambiguity in human thinking that fuzzy mathematics would prove more useful in the social sciences than in any other set of disciplines. We believe he is right. Indeed, the thesis of this short essay has been that the reason that it has not yet been more widely adopted is for the peculiar reason that the first efforts to employ fuzzy theory were applied to observable political phenomena. We believe that the application of fuzzy set theory to human preferences, which are mapped in spatial models, better suits the purposes for which fuzzy mathematics was intended. If we are right, then we will see a considerable literature on the subject in political science journals. In such case, the work presently appearing in fuzzy math journals [e.g., 25,26,27] will lay the foundation for empirical analyses appearing in future in political science journals. Of course, such analyses will require substantial investment in the development of fuzzy geometry [27] as well as measurement issues [28,29]. But that is the subject for another article.....

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Neutrosophic Relational Data Model

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Abstract. In this paper, we present a generalization of the relational data model based on interval neutrosophic set [1] . Our data model is capable of manipulating incomplete as well as inconsistent information. Fuzzy relation or intuitionistic fuzzy relation can only handle incomplete information. Associated with each relation are two membership functions one is called truth-membership function T which keeps track of the extent to which we believe the tuple is in the relation, another is called falsity-membership function F which keeps track of the extent to which we believe that it is not in the relation. A neutrosophic relation is inconsistent if there exists one tuple α such that $T(\alpha)+F(\alpha)>1$. In order to handle inconsistent situation, we propose an operator called "split" to transform inconsistent neutrosophic relations into pseudo-consistent neutrosophic relations and do the set-theoretic and relation-theoretic operations on them and finally use another operator called "combine" to transform the result back to neutrosophic relation. For this data model, we define algebraic operators that are generalizations of the usual operators such as intersection, union, selection, join on fuzzy relations. Our data model can underlie any database and knowledge-base management system that deals with incomplete and inconsistent information.

Keyword: Interval neutrosophic set, fuzzy relation, inconsistent information, incomplete information, neutrosophic relation.

1 Introduction

Relational data model was proposed by Ted Codd's pioneering paper [2] . Since then, relational database systems have been extensively studied and a lot of commercial relational database systems are currently available [3, 4]. This data model usually takes care of only well-defined and unambiguous data. However, imperfect information is ubiquitous – almost all the information that we have about the real world is not certain, complete and precise [5] . Imperfect information can be classified as: incompleteness, imprecision, uncertainty, and inconsistency. Incompleteness arises from the absence of a value, imprecision from the existence of a value which cannot be measured with suitable precision, uncertainty from the fact that a person has given a subjective opinion about the truth of a fact which he/she does not know for certain, and inconsistency from the fact that there are two or more conflicting values for a variable.

In order to represent and manipulate various forms of incomplete information in relational databases, several extensions of the classical relational model have been proposed [6, 7, 8, 9, 10, 11]. In some of these extensions, a variety of "null values" have been introduced to model unknown or not-applicable data values. Attempts have also been made to generalize operators of relational algebra to manipulate such extended data models [6, 8, 11, 12, 13]. The fuzzy set theory and fuzzy logic proposed by Zadeh [14] provide a requisite mathematical framework for dealing with incomplete and imprecise information. Later on, the concept of interval-valued fuzzy sets was proposed to capture the fuzziness of grade of membership itself [15] . In 1986, Atanassov introduced the intuitionistic fuzzy set [16] which is a generalization of fuzzy set and provably equivalent to interval-valued fuzzy set. The intuitionistic fuzzy sets consider both truth-membership T and falsity-membership F with $T(a), F(a) \in [0,1]$ and $T(a) + F(a) \leq 1$. Because of the restriction, the fuzzy set, interval-valued fuzzy set, and intuitionistic fuzzy set cannot handle inconsistent information. Some authors [17, 18, 19, 20, 21, 22, 23] have studied relational databases in the light of fuzzy set theory with an objective to accommodate a wider range of real-world requirements and to provide closer man-machine interactions. Probability, possibility, and Dempster-Shafer theory have been proposed to deal with uncertainty. Possibility theory [24] is built upon the idea of a fuzzy restriction. That means a variable could only take its value from some fuzzy set of values and any value within that set is a possible value for the variable. Because values have different degrees of membership in the set, they are possible to different degrees. Prade and Testemale [25] initially suggested using possibility theory to deal with incomplete and uncertain information in database. Their work is extended in [26] to cover multivalued attributes. Wong [27] proposes a method that quantifies the uncertainty in a database using probabilities. His method maybe is the simplest one which attached a probability to every member of a relation, and to use these values to provide the probability that a particular value is the correct answer to a particular query. Carvallo and Pittarelli [28] also use probability theory to model uncertainty in relational databases systems. Their method augmented projection and join operations with probability measures.

However, unlike incomplete, imprecise, and uncertain information, inconsistent information has not enjoyed enough research attention. In fact, inconsistent information exists in a lot of applications. For example, in data warehousing application, inconsistency will appear when trying to integrate the data from many different sources. Another example is that in the expert system, there exist facts which are inconsistent with each other. Generally, two basic approaches have been followed in solving the inconsistency problem in knowledge base: belief revision and paraconsistent logic. The goal of the first approach is to make an inconsistent theory consistent, either by revising it or by representing it by a consistent semantics. On the other hand, the paraconsistent approach allows reasoning in the presence of inconsistency, and contradictory information can be derived or introduced without trivialization [29] . Bagai and Sunderraman [30, 31] proposed a paraconsistent realational data model to deal with incomplete and inconsistent information. The data model has been applied to compute the well-founded and fitting model of logic programming [32, 33]. This

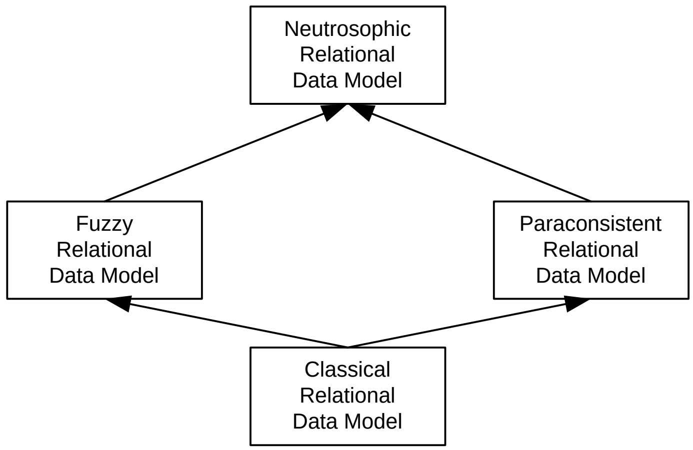

Fig. 1. Relationship among RDM, FRDM, PRDM, and NRDM

data model is based on paraconsistent logics which were studied in detail by de Costa [34] and Belnap [35] .

In this paper, we present a new relational data model − neutrosophic relational data model (NRDM). Our model is based on the neutrosophic set theory which is an extension of intuitionistic fuzzy set theory [36] and is capable of manipulating incomplete as well as inconsistent information. We use both truth-membership function grade α and falsity-membership function grade β to denote the status of a tuple of a certain relation with $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 2$. NRDM is the generalization of fuzzy relational data model (FRDM). That is , when $\alpha + \beta = 1$, neutroshophic relation is the ordinary fuzzy relation. This model is distinct with paraconsistent relational data model (PRDM), in fact it can be easily shown that PRDM is a special case of NRDM. That is, when α, $\beta = 0$ or 1 , neutrosophic relation is just paraconsistent relation. We can use Figure 1 to express the relationship among FRDM, PRDM, and NRDM.

We introduce neutrosophic relations, which are the fundamental mathematical structures underlying our model. These structures are strictly more general than classical fuzzy relations and intuititionistic fuzzy relations (interval-valued fuzzy relations), in that for any fuzzy relation or intuitionistic fuzzy relation there is a neutrosophic relation with the same information content, but not vice versa. The claim is also true for the relationship between neutrosophic relations and paraconsistent relations. We define algebraic operators over neutrosophic relations that extend the standard operators such as selection, join, union over fuzzy relations.

There are many potential applications of our new data model. Here are some examples:

1. Web mining. Essentially the data and documents on the Web are heterogeneous, inconsistency is unavoidable. Using the presentation and reasoning method of our data model, it is easier to capture imperfect information on the Web which will provide more potentially valued-added information.
2. Bioinformatics. There is a proliferation of data sources. Each research group and each new experimental technique seems to generate yet another source of valuable data. But these data can be incomplete and imprecise, and even inconsistent. We could not simply throw away one

data in favor of other data. So how to represent and extract useful information from these data will be a challenge problem.

3. Decision Support System. In decision support system, we need to combine the database with the knowledge base. There will be a lot of uncertain and inconsistent information, so we need an efficient data model to capture these information and reasoning with these information.

The paper is organized as follow. Section 2 deals with some of the basic definitions and concepts of fuzzy relations and operations. Section 3 introduces neutrosophic relations and two notions of generalizing the fuzzy relational operators such as union, join, projection for these relations. Section 4 presents some actual generalized algebraic operators for the neutrosophic relations. These operators can be used for specifying queries for database systems built on such relations. Section 5 gives an illustrative application of these operators. Finally, section 6 contains some concluding remarks and directions for future work.

2 Fuzzy Relations and Operations

In this section, we present the essential concepts of a fuzzy relational database. Fuzzy relations associate a value between 0 and 1 with every tuple representing the degree of membership of the tuple in the relation. We also present several useful query operators on fuzzy relations.

Let a relation scheme (or just scheme) Σ be a finite set of attribute names, where for any attribute name $A \in \Sigma$, dom(A) is a non-empty domain of values for A . A tuple on Σ is any map $t : \Sigma \to \bigcup_{A\in\Sigma} dom(A)$, such that $t(A) \in dom(A)$, for each $A \in \Sigma$. Let $\tau(\Sigma)$ denote the set of all tuples on Σ .

Definition 2.1 (Fuzzy relation). A fuzzy relation on scheme Σ is any map $R : \tau(\Sigma) \to [0,1]$. We let $F(\Sigma)$ be the set of all fuzzy relations on Σ .

If Σ and Δ are relation schemes such that $\Delta \subseteq \Sigma$, then for any tuple $t \in \tau(\Delta)$, we let t^{Σ} denote the set $\{t' \in \tau(\Sigma) \mid t'(A) = t(A), \text{ for all } A \in \Delta\}$ of all extensions of t . We extend this notion for any $T \subseteq \tau(\Delta)$ by defining $T^{\Sigma} = \bigcup_{t\in T} t^{\Sigma}$.

3 Set-theoretic operations on Fuzzy relations

Union Let R and S be fuzzy relations on scheme Σ . Then, $R \cup S$ is a fuzzy relation on scheme Σ given by

$$(R \cup S)(t) = \max\{R(t), S(t)\}, \tag{1}$$

for any $t \in \tau(\Sigma)$.

Definition 3.1 (Complement). Let R be a fuzzy relation on scheme Σ . Then, $-R$ is a fuzzy relation on scheme Σ given by

$$(-R)(t) = 1 - R(t), \tag{2}$$

for any $t \in \tau(\Sigma)$.

Definition 3.2 (Intersection). Let R and S be fuzzy relations on scheme Σ . Then $R \cap S$ is a fuzzy relation on scheme Σ given by

$$(R \cap S)(t) = \min\{R(t), S(t)\}, \tag{3}$$

for any $t \in \tau(\Sigma)$.

Definition 3.3 (Difference). Let R and S be fuzzy relations on scheme Σ . Then, $R - S$ is a fuzzy relation on scheme Σ given by

$(R - S)(t) = \min\{R(t), 1 - S(t)\}$, for any $t \in \tau(\Sigma)$.

4 Relation-theoretic operations on Fuzzy relations

Definition 4.1. Let R and S be fuzzy relations on schemes Σ and Δ , respectively. Then, the natural join (or just join) of R and S , denoted $R \infty S$ is a fuzzy relation on scheme $\Sigma \cup \Delta$, given by

$$(R \infty S)(t) = \min\{R(\pi_\Sigma(t)), S(\pi_\Delta(t))\}, \tag{4}$$

for any $t \in \tau(\Sigma \cup \Delta)$.

Definition 4.2. Let R be a fuzzy relation on scheme Σ and let $\Delta \subseteq \Sigma$. Then, the projection of R onto Δ , denoted by $\prod_\Delta(R)$ is a fuzzy relation on scheme Δ given by

$$(\prod_\Delta(R))(t) = \max\{R(u) \mid u \in t^\Sigma\}, \tag{5}$$

for any $t \in \tau(\Delta)$.

Definition 4.3. Let R be a fuzzy relation on scheme Σ , and let F be any logic formula involving attribute names in Σ , constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the selection of R by F , denoted $\dot{\sigma}_F(R)$, is a fuzzy relation on scheme Σ , given by

$$(\dot{\sigma}_F(R))(t) = \begin{cases} R(t) if t \in \sigma_F(\tau(\Sigma)) \\ 0 otherwise \end{cases}$$

where σ_F is the usual selection of tuples satisfying F .

5 Neutrosophic Relations

In this section, we generalize fuzzy relations in such a manner that we are now able to assign a measure of belief and a measure of doubt to each tuple. We shall refer to these generalized fuzzy relations as neutrosophic relations. So, a tuple in a neutrosophic relation is assigned a measure $\langle \alpha, \beta \rangle, 0 \le \alpha, \beta \le 1$. α will be referred to as the belief factor and β will be referred to as the doubt factor. The interpretation of this measure is that we believe with confidence α and doubt with confidence β that the tuple is in the relation. The belief and doubt confidence factors for a tuple need not add to exactly 1. This allows for incompleteness and inconsistency to be represented. If the belief and doubt factors add up to less than 1, we have incomplete information regarding the tuple's status in the relation and if the belief and doubt factors add up to more than 1, we have inconsistent information regarding the tuple's status in the relation.

In contrast to fuzzy relations where the grade of membership of a tuple is fixed, neutrosophic relations bound the grade of membership of a tuple to a subinterval $[\alpha, 1-\beta]$ for the case $\alpha+\beta \leq 1$.

The operators on fuzzy relations can also be generalized for neutrosophic relations. However, any such generalization of operators should maintain the belief system intuition behind neutrosophic relations.

This section also develops two different notions of operator generalizations.

We now formalize the notion of a neutrosophic relation.

Recall that $\tau(\Sigma)$ denotes the set of all tuples on any scheme Σ .

Definition 9 A neutrosophic relation R on scheme Σ is any subset of

$$\tau(\Sigma) \times [0,1] \times [0,1]$$

For any $t \in \tau(\Sigma)$, we shall denote an element of R as $\langle t, R(t)^+, R(t)^- \rangle$, where $R(t)^+$ is the belief factor assigned to t by R and $R(t)^-$ is the doubt factor assigned to t by R . Let $V(\Sigma)$ be the set of all neutrosophic relations on Σ .

Definition 10 A neutrosophic relation R on scheme Σ is consistent if $R(t)^+ + R(t)^- \leq 1$, for all $t \in \tau(\Sigma)$. Let $C(\Sigma)$ be the set of all consistent neutrosophic relations on Σ . R is said to be complete if $R(t)^+ + R(t)^- \geq 1$, for all $t \in \tau(\Sigma)$. If R is both consistent and complete, i.e. $R(t)^+ + R(t)^- = 1$, for all $t \in \tau(\Sigma)$, then it is a total neutrosophic relation, and let $T(\Sigma)$ be the set of all total neutrosophic relations on Σ .

Definition 5.1. R is said to be pseudo-consistent if

$$\max\{b_i \mid (\exists t \in \tau(\Sigma))(\exists d_i)(\langle t, b_i, d_i \rangle \in R)\} + \tag{6}$$
$$\max\{d_i \mid (\exists t \in \tau(\Sigma))(\exists b_i)(\langle t, b_i, d_i \rangle \in R\} > 1 \ , \tag{7}$$

where for these $\langle t, b_i, d_i \rangle, b_i + d_i = 1$. Let $P(\Sigma)$ be the set of all pseudo-consistent neutrosophic relations on Σ .

Example 5.1. Neutrosophic relation $R = \{\langle a, 0.3, 0.7 \rangle, \langle a, 0.4, 0.6 \rangle, \langle b, 0.2, 0.5 \rangle, \langle c, 0.4, 0.3 \rangle\}$ is pseudo-consistent. Because for $t = a$, $\max\{0.3, 0.4\} + \max\{0.7, 0.6\} = 1.1 > 1$.

It should be observed that total neutrosophic relations are essentially fuzzy relations where the uncertainty in the grade of membership is eliminated. We make this relationship explicit by defining a one-one correspondence $\lambda_\Sigma : T(\Sigma) \rightarrow F(\Sigma)$, given by $\lambda_\Sigma(R)(t) = R(t)^+$, for all $t \in \tau(\Sigma)$. This correspondence is used frequently in the following discussion.

5.1 Operator Generalizations

It is easily seen that neutrosophic relations are a generalization of fuzzy relations, in that for each fuzzy relation there is a neutrosophic relation with the same information content, but not vice versa. It is thus natural to think of generalizing the operations on fuzzy relations such as union, join, and projection etc. to neutrosophic relations. However, any such generalization should be intuitive with respect to the belief system model of neutrosophic relations. We now construct a framework for operators on both kinds of relations and introduce two different notions of the generalization relationship among their operators.

An n -ary operator on fuzzy relations with signature $\langle \Sigma, ..., \Sigma \rangle$ is a function $\Theta : F(\Sigma) \times \cdots \times F(\Sigma) \rightarrow F(\Sigma)$, where $\Sigma, ...\Sigma$ are any schemes. Similarly, an n - ary operator on neutrosophic relations with signature $\langle \Sigma, ..., \Sigma \rangle$ is a function $\Psi : V(\Sigma) \times \cdots V(\Sigma) \rightarrow V(\Sigma)$.

Definition 12 An operator Ψ on neutrosophic relations with signature $\langle \Sigma, ..., \Sigma \rangle$ is totality preserving if for any total neutrosophic relations $R_1, ..., R_n$ on schemes

Definition 5.2. A totality preserving operator Ψ on neutrosophic relations with signature $\langle \Sigma, ..., \Sigma \rangle$ is a weak generalization of an operator Θ on fuzzy relations with the same signature, if for any total neutrosophic relations $R_1, ..., R_n$ on scheme

$$\lambda_\Sigma(\Psi(R_1, ..., R_n)) = \Theta(\lambda_\Sigma(R_1), ..., \lambda_\Sigma(R_n)) \ .$$

The above definition essentially requires Ψ to coincide with Θ on total neutrosophic realtions (which are in one-one correspondence with the fuzzy relations). In general, there may be many operators on neutrosophic relations that are weak generalizations of a given operator Θ on fuzzy relations. The behavior of the weak generalizations of Θ on even just the consistent neutrosophic relations may in general vary. We require a stronger notion of operator generalization under which, at least when restricted to consistent neutrosophic relations, the behavior of all the generalized operators is the same. Before we can develop such a notion, we need that of 'representation' of a neutrosophic relation.

We associate with a consistent neutrosophic relation R the set of all (fuzzy relations corresponding to) total neutrosophic relations obtainable from R by filling the gaps between the belief and doubt factors for each tuple. Let the map $reps_\Sigma : C(\Sigma) \rightarrow 2^{F(\Sigma)}$ be given by

$$reps_\Sigma(R) = \{Q \in F(\Sigma) \mid \bigwedge_{t_i \in \tau(\Sigma)} (R(t_i)^+ \leq Q(t_i) \leq 1 - R(t_i)^-)\} \ .$$

The set $reps_\Sigma(R)$ contains all fuzzy relations that are 'completions' of the consistent neutrosophic relation R . Observe that $reps_\Sigma$ is defined only for consistent neutrosophic relations and produces sets of fuzzy relations. Then we have following observation.

Proposition 5.1. For any consistent neutrosophic relation R on scheme Σ , $reps_\Sigma(R)$ is the singleton $\{\lambda_\Sigma(R)\}$ iff R is total.

Proof. It is clear from the definition of consistent and total neutrosophic relations and from the definition of *reps* operation.

We now need to extend operators on fuzzy relations to sets of fuzzy relations. For any operator $\Theta : F(\Sigma) \cdots F(\Sigma) \rightarrow F(\Sigma)$ on fuzzy relations, we let $S(\Theta) : 2^{F(\Sigma)} \times \cdots \times 2^{F(\Sigma)} \rightarrow 2^{F(\Sigma)}$ be a map on sets of fuzzy relations defined as follows. For any sets $M_{1,...,}M_n$ of fuzzy relations on schemes $\Sigma, ..., \Sigma$, respectively,

$$S(\Theta)(M_1, ..., M_n) = \{\Theta(R_1, ..., R_n) \mid R_i \in M_i, \tag{8}$$

for all $i, 1 \leq i \leq n\}$.

In other words, $S(\Theta)(M_1, ..., M_n)$ is the set of Θ - images of all tuples in the Cartesian product $M_1 \times \cdots \times M_n$. We are now ready to lead up to a stronger notion of operator generalization.

Definition 5.3. An operator Ψ on neutrosophic relations with signature $\langle \Sigma, ..., \Sigma \rangle$ is consistency preserving if for any consistent neutrosophic relations $R_1, ..., R_n$ on schemes $\Sigma, ..., \Sigma$, respectively, $\Psi(R_1, ..., R_n)$ is also consistent.

Definition 5.4. A consistency preserving operator Ψ on neutrosophic relations with signature $\langle \Sigma, ..., \Sigma \rangle$ is a strong generalization of an operator Θ on fuzzy relations with the same signature, if for any consistent neutrosophic relations $R_1, ..., R_n$ on schemes $\Sigma, ..., \Sigma$, respectively, we have

$$reps_\Sigma(\Psi(R_1, ..., R_n)) = S(\Theta)(reps_\Sigma(R_1), ..., reps_\Sigma(R_n)). \tag{9}$$

Given an operator Θ on fuzzy relations, the behavior of a weak generalization of Θ is 'controlled' only over the total neutrosophic relations. On the other hand, the behavior of a strong generalization is 'controlled' over all consistent neutrosophic relations. This itself suggests that strong generalization is a stronger notion than weak generalization. The following proposition makes this precise.

Proposition 5.2. If Ψ is a strong generalization of Θ , then Ψ is also a weak generalization of Θ .

Proof. Let $\langle \Sigma, ..., \Sigma \rangle$ be the signature of Ψ and Θ , and let $R_1, ..., R_n$ be any total neutrosophic relations on schemes $\Sigma, ..., \Sigma$, respectively. Since all total relations are consistent, and Ψ is a strong generalization of Θ , we have that

$$reps_\Sigma(\Psi(R_1, ..., R_n)) = S(\Theta)(reps_\Sigma(R_1), ..., reps_\Sigma(R_n)), \tag{10}$$

Proposition 5.1 gives us that for each $i, 1 \le i \le n$, $reps_\Sigma(R_i)$ is the singleton set $\{\lambda_\Sigma(R_i)\}$. Therefore, $S(\Theta)(reps_\Sigma(R_i), ..., reps_\Sigma(R_n))$ is just the singleton set: $\{\Theta(\lambda_\Sigma(R_1), ..., \lambda_\Sigma(R_n))\}$. Here, $\Psi(R_1, ..., R_n)$ is total, and $\lambda_\Sigma(\Psi(R_1, ..., R_n)) = \Theta(\lambda_\Sigma(R_1), ..., \lambda_\Sigma(R_n))$, i.e. Ψ is a weak generalization of Θ .

Though there may be many strong generalizations of an operator on fuzzy relations, they all behave the same when restricted to consistent neutrosophic relations. In the next section, we propose strong generalizations for the usual operators on fuzzy relations. The proposed generalized operators on neutrosophic relations correspond to the belief system intuition behind neutrosophic relations.

First we will introduce two special operators on neutrosophic relations called split and combine to transform inconsistent neutrosophic relations into pseudo-consistent neutrosophic relations and transform pseudo-consistent neutrosophic relations into inconsistent neutrosophic relations.

Definition 5.5 (Split Operator Δ). Let R be a neutrosophic relation on scheme Σ . Then,

$$\Delta(R) = \{\langle t, b, d \rangle \mid \langle t, b, d \rangle \in R and b + d \le 1\} \cup \tag{11}$$

$$\{\langle t, b', d' \rangle \mid \langle t, b, d \rangle \in R and b + d > 1 and b' = b and d' = 1 - b\} \cup \tag{12}$$

$$\{\langle t, b', d' \rangle \mid \langle t, b, d \rangle \in R and b + d > 1 b' = 1 - d and d' = d\} \ . \tag{13}$$

It is obvious that $\Delta(R)$ is pseudo-consistent if R is inconsistent.

Definition 5.6 (Combine Operator ∇). Let R be a neutrosophic relation on scheme Σ . Then,

$$\nabla(R) = \{\langle t, b', d' \rangle \mid (\exists b)(\exists d)((\langle t, b', d \rangle \in R and (\forall b_i, d_i)(\langle t, b_i, d_i \rangle \to b' \ge b_i) and \tag{14}$$

$$\langle t, b, d' \rangle \in R and (\forall b_i)(\forall d_i)(\langle t, b_i, d_i \rangle \to d' \ge d_i))\}. \tag{15}$$

It is obvious that $\nabla(R)$ is inconsistent if R is pseudo-consistent.

Note that strong generalization defined above only holds for consistent or pseudo-consistent neutrosophic relations. For any arbitrary neutrosophic relations, we should first use split operation to transform them into non-inconsistent neutrosophic relations and apply the set-theoretic and relation-theoretic operations on them and finally use combine operation to transform the result into arbitrary neutrosophic relation. For the simplification of notation, the following generalized algebra is defined under such assumption.

6 Generalized Algebra on Neutrosophic Relations

In this section, we present one strong generalization each for the fuzzy relation operators such as union, join, and projection. To reflect generalization, a hat is placed over a fuzzy relation operator to obtain the corresponding neutrosophic relation operator. For example, ∞ denotes the natural join among fuzzy relations, and $\hat{\infty}$ denotes natural join on neutrosophic relations. These generalized operators maintain the belief system intuition behind neutrosophic relations.

7 Set-Theoretic Operators

We first generalize the two fundamental set-theoretic operators, union and complement.

Definition 7.1. Let R and S be neutrosophic relations on scheme Σ . Then,

1. the union of R and S , denoted $R\hat{\cup}S$, is a neutrosophic relation on scheme Σ , given by
$$(R\hat{\cup}S)(t) = \left\langle \max\{R(t)^+, S(t)^+\}, \min\{R(t)^-, S(t)^-\} \right\rangle, \tag{16}$$
for any $t \in \tau(\Sigma)$;
2. the complement of R , denoted $\hat{-}R$, is a neutrosophic relation on scheme Σ , given by
$$(\hat{-}R)(t) = \left\langle R(t)^-, R(t)^+ \right\rangle, \tag{17}$$
for any $t \in \tau(\Sigma)$.

An intuitive appreciation of the union operator can be obtained as follows: Given a tuple t , since we believed that it is present in the relation R with confidence $R(t)^+$ and that it is present in the relation S with confidence $S(t)^+$, we can now believe that the tuple t is present in the "either $-$ R $-$ or $-$ S " relation with confidence which is equal to the larger of $R(t)^+$ and $S(t)^+$. Using the same logic, we can now believe in the absence of the tuple t from the "either $-$ R $-$ or $-$ S " relation with confidence which is equal to the smaller (because t must be absent from both R and S for it to be absent from the union) of $R(t)^-$ and $S(t)^-$. The definition of complement and of all the other operators on neutrosophic relations defined later can (and should) be understood in the same way.

Proposition 7.1. The operators $\hat{\cup}$ and unary $\hat{-}$ on neutrosophic relations are strong generalizations of the operators $\cup$ and unary $-$ on fuzzy relations.

Proof. Let R and S be consistent neutrosophic relations on scheme Σ . Then $reps_\Sigma(R\hat{\cup}S)$ is the set

$$\{Q \mid \bigwedge_{t_i \in \tau(\Sigma)} (\max\{R(t_i)^+, S(t_i)^+\} \leq Q(t_i) \leq 1 - \min\{R(t_i)^-, S(t_i)^-\})\}$$

This set is the same as the set

$$\{r \cup s \mid \bigwedge_{t_i \in \tau(\Sigma)} (R(t_i)^+ \leq r(t_i) \leq 1 - R(t_i)^-), \bigwedge_{t_i \in \tau(\Sigma)} (S(t_i)^+ \leq s(t_i) \leq 1 - S(t_i)^-)\}$$

which is $S(\cup)(reps_\Sigma(R), reps_\Sigma(S))$. Such a result for unary $\hat{-}$ can also be shown similarly.

For sake of completeness, we define the following two related set-theoretic operators:

Definition 7.2. Let R and S be neutrosophic relations on scheme Σ . Then,

1. the intersection of R and S , denoted $R\hat{\cap}S$, is a neutrosophic relation on scheme Σ , given by

$$(R\hat{\cap}S)(t) = \left\langle \min\{R(t)^+, S(t)^+\}, \max\{R(t)^-, S(t)^-\} \right\rangle, \tag{18}$$

for any $t \in \tau(\Sigma)$.

2. the difference of R and S , denoted $R\hat{-}S$, is a neutrosophic relation on scheme Σ , given by

$$(R\hat{-}S)(t) = \left\langle \min\{R(t)^+, S(t)^-\}, \max\{R(t)^-, S(t)^+\} \right\rangle, \tag{19}$$

for any $t \in \tau(\Sigma)$.

The following proposition relates the intersection and difference operators in terms of the more fundamental set-theoretic operators union and complement.

Proposition 7.2. For any neutrosophic relations R and S on the same scheme, we have

$$R\hat{\cap}S = \hat{-}(\hat{-}R\hat{\cup}\hat{-}S), \tag{20}$$

and

$$R\hat{-}S = \hat{-}(\hat{-}R\hat{\cup}S). \tag{21}$$

Proof. By definition,

$$\hat{-}R(t) = \left\langle R(t)^-, R(t)^+ \right\rangle \tag{22}$$

$$\hat{-}S(t) = \left\langle S(t)^-, S(t)^+ \right\rangle \tag{23}$$

and

$$(\hat{-}R\hat{\cup}\hat{-}S)(t) = \left\langle \max(R(t)^-, S(t)^-), \min(R(t)^+, S(t)^+) \right\rangle \tag{24}$$

so

$$(\hat{-}(\hat{-}R\hat{\cup}\hat{-}S))(t) = \left\langle \min(R(t)^+, S(t)^+), \max(R(t)^-, S(t)^-) \right\rangle = R\hat{\cap}S(t). \tag{25}$$

The second part of the result can be shown similarly.

8 Relation-Theoretic Operators

We now define some relation-theoretic algebraic operators on neutrosophic relations.

Definition 8.1. Let R and S be neutrosophic relations on schemes Σ and Δ , respectively. Then, the natural join (further for short called join) of R and S , denoted $R\hat{\infty}S$, is a neutrosophic relation on scheme $\Sigma \cup \Delta$, given by

$$(R \hat{\infty} S)(t) = \left\langle \min\{R(\pi_\Sigma(t))^+, S(\pi_\Delta(t))^+\}, \max\{R(\pi_\Sigma(t))^-, S(\pi_\Delta(t))^-\} \right\rangle,$$

where π is the usual projection of a tuple.

It is instructive to observe that, similar to the intersection operator, the minimum of the belief factors and the maximum of the doubt factors are used in the definition of the join operation.

Proposition 8.1. $\hat{\infty}$ is a strong generalization of ∞ .

Proof. Let R and S be consistent neutrosophic relations on schemes Σ and Δ , respectively. Then $reps_{\Sigma\cup\Delta}(R \hat{\infty} S)$ is the set $\{Q \in F(\Sigma\cup\Delta) \mid \wedge_{t_i\in\tau(\Sigma\cup\Delta)}(\min\{R_{\pi_\Sigma}(t_i)^+, S_{\pi_\Delta}(t_i)^+\} \leq Q(t_i) \leq 1-\max\{R_{\pi_\Sigma}(t_i)^-, S_{\pi_\Delta}(t_i)^-\})\}$ and $S(\infty)(reps_\Sigma(R), reps_\Delta(S)) = \{r \infty S \mid r \in reps\Sigma(R), s \in reps_\Delta(S)\}$.

Let $Q \in reps_{\Sigma\cup\Delta}(R \hat{\infty} S)$. Then $\pi_\Sigma(Q) \in reps\Sigma(R)$, where π_Σ is the usual projection over Σ of fuzzy relations. Similarly, $\pi_\Delta(Q) \in reps_\Delta(R)$, Therefore, $Q \in S(\infty)(reps_\Sigma(R), reps_\Delta(S))$.

Let $Q \in S(\infty)(reps\Sigma(R), reps_\Delta(S))$. Then $Q(t_i) \geq \min\{R_{\pi_\Sigma}(t_i)^+, S_{\pi_\Delta}(t_i)^+\}$ and $Q(t_i) \leq \min\{1-R_{\pi_\Sigma}(t_i)^-, 1-S\pi_\Delta(t_i)^-\} = 1-\max\{R_{\pi_\Sigma}(t_{i,})^-, S_{\pi_\Delta}(t_i)^-\}$, for any $t_i \in \tau(\Sigma\cup\Delta)$, because R and S are consistent. Therefore, $Q \in reps_{\Sigma\cup\Delta}(R \hat{\infty} S)$.

We now present the projection operator.

Definition 8.2. Let R be a neutrosophic relation on scheme Σ , and $\Delta \subseteq \Sigma$. Then, the projection of R onto Δ , denoted $\hat{\pi}(R)$, is a neutrosophic relation on scheme Δ , given by

$$(\hat{\pi}(R))(t) = \left\langle \max\{R(u)^+ \mid u \in t^\Sigma\}, \min\{R(u)^- \mid u \in t^\Sigma\} \right\rangle.$$

The belief factor of a tuple in the projection is the maximum of the belief factors of all of the tuple's extensions onto the scheme of the input neutrosophic relation. Moreover, the doubt factor of a tuple in the projection is the minimum of the doubt factors of all of the tuple's extensions onto the scheme of the input neutrosophic relation.

We present the selection operator next.

Definition 8.3. Let R be a neutrosophic relation on scheme Σ , and let F be any logic formula involving attribute names in Σ , constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the selection of R by F , denoted $\hat{\sigma}_F(R)$, is a neutrosophic relation on scheme Σ , given by

$$(\hat{\sigma}_F(R))(t) = \langle \alpha, \beta \rangle, \tag{26}$$

where

$$\alpha = \begin{cases} R(t)^+ \ if \ t \in \sigma_F(\tau(\Sigma)) \\ 0 \qquad otherwise \end{cases} \tag{27}$$

and

$$\beta = \begin{cases} R(t)^- \ if \ t \in \sigma_F(\tau(\Sigma)) \\ 0 \qquad otherwise \end{cases} \tag{28}$$

where σ_F is the usual selection of tuples satisfying F from ordinary relations.

If a tuple satisfies the selection criterion, its belief and doubt factors are the same in the selection as in the input neutrosophic relation. In the case where the tuple does not satisfy the selection criterion, its belief factor is set to 0 and the doubt factor is set to 1 in the selection.

Proposition 8.2. The operators $\hat{\pi}$ and $\hat{\sigma}$ are strong generalizations of π and σ , respectively.

Proof. Similar to that of Proposition 8.1.

Example 8.1. Relation schemes are sets of attribute names, but in this example we treat them as ordered sequence of attribute names (which can be obtained through permutation of attribute names), so tuples can be viewed as the usual lists of values. Let $\{a,b,c\}$ be a common domain for all attribute names, and let R and S be the following neutrosophic relations on schemes $\langle X,Y\rangle$ and $\langle Y,Z\rangle$ respectively.

t	**R(t)**
(a,a)	< 0,1 >
(a,b)	< 0,1 >
(a,c)	< 0,1 >
(b,b)	< 1,0 >
(b,c)	< 1,0 >
(c,b)	< 1,1 >

t	**S(t)**
(a,c)	< 1,0 >
(b,a)	< 1,1 >
(c,b)	< 0,1 >

Table 1: The neutrosophic relations R and S.

For other tuples which are not in the neutrosophic relations $R(t)$ and $S(t)$, their $\langle\alpha,\beta\rangle = \langle 0,0\rangle$ which means no any information available. Because R and S are inconsistent, we first use split operation to transform them into pseudo-consistent and apply the relation-theoretic operations on them and transform the result back to arbitrary neutrosophic set using combine operation. Then, $T_1 = \nabla(\Delta(R) \hat{\infty} \Delta(S))$ is a neutrosophic relation on scheme $\langle X,Y,Z\rangle$ and $T_2 = \nabla(\hat{\pi}(\Delta(T_1)))$ and $T_3 = \hat{\sigma}_{X\neg=Z}(T_2)$ are neutrosophic relations on scheme $\langle X,Z\rangle$. T_1 , T_2 , and T_3 are shown below:

t	**T_1(t)**
(a,a,a)	< 0,1 >
(a,a,b)	< 0,1 >
(a,a,c)	< 0,1 >
(a,b,a)	< 0,1 >
(a,b,b)	< 0,1 >
(a,b,c)	< 0,1 >
(a,c,a)	< 0,1 >
(a,c,b)	< 0,1 >
(a,c,c)	< 0,1 >
(b,b,a)	< 1,1 >
(b,c,b)	< 0,1 >
(c,b,a)	< 1,1 >
(c,b,b)	< 0,1 >
(c,b,c)	< 0,1 >
(c,c,b)	(< 0,1 >

t	**T_2(t)**
(a,a)	< 0,1 >
(a,b)	< 0,1 >
(a,c)	< 0,1 >
(b,a)	< 1,0 >
(c,a)	< 1,0 >

t	**T_3(t)**
(a,a)	< 0,1 >
(a,b)	< 0,1 >
(a,c)	< 0,1 >
(b,a)	< 1,0 >
(b,b)	< 0,1 >
(c,a)	< 1,0 >
(c,c)	< 0,1 >

Table 2: The neutrosophic relations T_1, T_2, and T_3.

9 An Application

Consider the target recognition example presented in [36] . Here, an autonomous vehicle needs to identify objects in a hostile environment such as a military battlefield. The autonomous vehicle is equipped with a number of sensors which are used to collect data, such as speed and size of the objects (tanks) in the battlefield. Associated with each sensor, we have a set of rules that describe the type of the object based on the properties detected by the sensor.

Let us assume that the autonomous vehicle is equipped with three sensors resulting in data collected about radar readings, of the tanks, their gun characteristics, and their speeds. What follows is a set of rules that associate the type of object with various observations.

Radar Readings:

- Reading r_1 indicates that the object is a T-72 tank with belief factor 0.80 and doubt factor 0.15.
- Reading r_2 indicates that the object is a T-60 tank with belief factor 0.70 and doubt factor 0.20.
- Reading r_3 indicates that the object is not a T-72 tank with belief factor 0.95 and doubt factor 0.05.
- Reading r_4 indicates that the object is a T-80 tank with belief factor 0.85 and doubt factor 0.10.

Gun Characteristics:

- Characteristic c_1 indicates that the object is a T-60 tank with belief factor 0.80 and doubt factor 0.20.
- Characteristic c_2 indicates that the object is not a T-80 tank with belief factor 0.90 and doubt factor 0.05.
- Characteristic c_3 indicates that the object is a T-72 tank with belief factor 0.85 and doubt factor 0.10.

Speed Characteristics:

- Low speed indicates that the object is a T-60 tank with belief factor 0.80 and doubt factor 0.15.
- High speed indicates that the object is not a T-72 tank with belief factor 0.85 and doubt factor 0.15.
- High speed indicates that the object is not a T-80 tank with belief factor 0.95 and doubt factor 0.05.
- Medium speed indicates that the object is not a T-80 tank with belief factor 0.80 and doubt factor 0.10.

These rules can be captured in the following three neutrosophic relations:

Reading	Object	Confidence Factors
r_1	T-72	$< 0.80,0.15 >$
r_2	T-60	$< 0.70,0.20 >$
r_3	T-72	$< 0.05,0.95 >$
r_4	T-80	$< 0.85,0.10 >$

Reading	Object	Confidence Factors
c_1	T-60	$< 0.80,0.20 >$
c_2	T-80	$< 0.05,0.90 >$
c_3	T-72	$< 0.85,0.10 >$

Reading	Object	Confidence Factors
low	T-60	$< 0.80,0.15 >$
high	T-72	$< 0.15,0.85 >$
high	T-80	$< 0.05,0.95 >$
medium	T-80	$< 0.10,0.80 >$

Table 3: Rules

The autonomous vehicle uses the sensors to make observations about the different objects and then uses the rules to determine the type of each object in the battlefield. It is quite possible that two different sensors may identify the same object as of different types, thereby introducing inconsistencies.

Let us now consider three objects o_1 , o_2 and o_3 which need to be identified by the autonomous vehicle. Let us assume the following observations made by the three sensors about the three objects. Once again, we assume certainty factors (maybe derived from the accuracy of the sensors) are associated with each observation.

Object-id	Reading	Confidence Factors
o_1	r_3	$< 1.00,0.00 >$
o_2	r_1	$< 1.00,0.00 >$
o_3	r_4	$< 1.00,0.00 >$

Object-id	Reading	Confidence Factors
o_1	c_3	$< 0.80,0.10 >$
o_2	c_1	$< 0.90,0.10 >$
o_3	c_2	$< 0.90,0.10 >$

Object-id	Reading	Confidence Factors
o_1	high	$< 0.90,0.10 >$
o_2	low	$< 0.95,0.05 >$
o_3	medium	$< 0.80,0.20 >$

Table 4: Data

Given these observations and the rules, we can use the following algebraic expression to identify the three objects:

$$\hat{\pi}(RadarData \hat{\bowtie} RadarRules)\hat{\cap}$$
$$\hat{\pi}(GunData \hat{\bowtie} GunRules)\hat{\cap}$$
$$\hat{\pi}(SpeedData \hat{\bowtie} SpeedRules)$$

The intuition behind the intersection is that we would like to capture the common (intersecting) information among the three sensor data. Evaluating this expression, we get the following neutrosophic relation:

Object-id	Reading	Confidence Factors
o_1	T-72	$< 0.05,0.00 >$
o_2	T-80	$< 0.00,0.05 >$
o_3	T-80	$< 0.05,0.00 >$

Table 5: Radar. Guns, and Speed relation

It is clear from the result that by the given information, we could not infer any useful information that is we could not decide the status of objects o_1 , o_2 and o_3 .

10 Conclusions and Future Work

We have presented a generalization of fuzzy relations, intuitionistic fuzzy relations (interval-valued fuzzy relations), and paraconsistent relations, called neutrosophic relations, in which we allow the representation of confidence (belief and doubt) factors with each tuple. The algebra on fuzzy relations is appropriately generalized to manipulate neutrosophic relations.

Various possibilities exist for further study in this area. Recently, there has been some work in extending logic programs to involve quantitative paraconsistency. Paraconsistent logic programs were introduced in [37] and probabilistic logic programs in [38] . Paraconsistent logic programs allow negative atoms to appear in the head of clauses (thereby resulting in the possibility of dealing with inconsistency), and probabilistic logic programs associate confidence measures with literals and with entire clauses. The semantics of these extensions of logic programs have already been presented, but implementation strategies to answer queries have not been discussed. We propose to use the model introduced in this paper in computing the semantics of these extensions of logic programs. Exploring application areas is another important thrust of our research.

We developed two notions of generalizing operators on fuzzy relations for neutrosophic relations. Of these, the stronger notion guarantees that any generalized operator is "well-behaved" for neutrosophic relation operands that contain consistent information.

For some well-known operators on fuzzy relations, such as union, join, and projection, we introduced generalized operators on neutrosophic relations. These generalized operators maintain the belief system intuition behind neutrosophic relations, and are shown to be "well-behaved" in the sense mentioned above.

Our data model can be used to represent relational information that may be incomplete and inconsistent. As usual, the algebraic operators can be used to construct queries to any database systems for retrieving vague information.

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Consistency degrees of theories and graded reasoning in n-valued Lukasiewicz propositional logic

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Abstract. [3]
In order to cope with the vagueness of human reasoning, many graded approaches were proposed in the past several years. In the present paper we introduce different approaches for graded reasoning in the framework of n-valued Łukasiewicz propositional logic $Ł_n$. As a special case of graded reasoning, we firstly introduce the concept of consistency degree of theories over $Ł_n$ to measure the extent to which a theory over $Ł_n$ is consistent. Secondly we propose several methods of general graded reasoning in $Ł_n$ of which comparison is obtained. Finally we introduce syntactically as well as semantically a new complete method of graded reasoning in $Ł_n$.

Key words– n-valued Łukasiewicz propositional logic, Theory, Deduction theorem, Completeness theorem, Satisfiability degree, Consistency degree, Degree of entailment, Graded reasoning

1 Introduction

"Black and white" reasoning is not adequate to cope with the essential vagueness of human reasoning which is approximate rather than precise in nature. Hence, the logical treatment of the concepts of vagueness and uncertainty is of increasing importance in artificial intelligence and related research areas. Consequently, many logicians have proposed different deductive systems of many-valued logic as a formalization of approximation (see, e.g., [1,2,4,9,8,14,20]). As far as we know, the proposals in [2,4,14] are obtained by enlarging only the range of truth degrees of formulas in which logical deduction is still exact and to make a consequence the antecedent clause of its rule must match its premise exactly, while in [9,8] a graded approach towards a general principle of the human mind was proposed where even the logical axioms and logical reasoning were all graded. But the approach seems so random that every formula even every axiom can arbitrarily receive a real number in the unit interval as its syntactic valuation. In this paper we propose different approaches

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for graded reasoning in which every formula is also equipped with a real number. The difference lies in that the real number associated to each formula is uniquely determined by the logical structure of the formula, and hence our approach will be different from that given in [9,8].

As a special case of graded reasoning in logic systems, the problem on the consistency degrees of theories (i.e., how to measure the extent to which the contradiction is a consequence of a given theory) is also one of the crucial questions and is not easy to be done yet. For trying to grade the extent of consistency of different theories, the authors of [8,3] firstly introduced the concept of inconsistency degree of a fuzzy theory and proved the sufficient and necessary condition for such a theory to be consistent. But such a result is not very ideal and reasonable just as the authors of [8] pointed out: "Later on, we will see that even the attempt to introduce some kind of degrees of consistency mostly does not work" (see P129). Fortunately, in [19,24,21,22], the authors, from logical point of view and based on deduction theorems, completeness theorems and the concept of satisfiability degrees of formulas, introduced, in classical and fuzzy propositional logic systems, a more natural and reasonable definition of consistency degrees of theories. In other words, we have studied the consistency of theories where the set of truth degrees jumped from $\{0,1\}$ to $[0,1]$. A natural question then arises: how to harmoniously fill in the gap of consistency of theories between $\{0,1\}$-valued logic and [0,1]-valued logic? That is to say, how to define the concept of consistency degrees of theories in n-valued logic system such that it approximates the consistency of theories in fuzzy logic system when n turns to infinity and takes the classical case as a special case when $n=2$?

As one has seen from [21,22], the methods of [21,22] consider only standard semantics of the logic systems and hence can be applied to formal theories over standard complete logic systems. Moreover, the concept of satisfiability degrees of formulas is a crucial tool for the analysis. Whence, in order to extend such results of [21,22] to some n-valued logic system, the n-valued logic system has to satisfy the following conditions:

i. The set of truth degrees should be a finite subset of [0,1],
ii. The n-valued logic system should be complete w.r.t the set of truth degrees above,
iii. The concept of satisfiability degrees of formulas in the n-valued logic system should be established which builds a bridge between $\{0,1\}$-valued logic and $[0,1]$-valued logic, i.e., the satisfiability degree function τ_n induced by satisfiability degrees should converge to the integrated satisfiability degree function τ as n converges to infinity and takes the classical theory as a special case when $n=2$.

In [23], we adapted the results of [22] to the n-valued R_0-logic (more precisely, the n-valued NM-logic) and answered the above question. Following the train of thought from special case to general, the present paper has two purposes. The first one is to consider the consistency degrees of theories and to adapt the results of [22] to $Ł_n$. The second is to consider the general reasoning, to propose several other methods of graded reasoning, and further to compare these results. As we will see, the methods of graded reasoning given in the present paper in which the satisfiability degrees of formulas is a crucial tool are different and far from those given in [8,9]. Most importantly we give syntactically as well as semantically a new complete method of graded reasoning in $Ł_n$.

The present paper is structured as follows: In order to make the paper as self-contained as possible, we remind in Section 2 the representation of $Ł_n$ and the basic notions that will be used throughout the paper, and review the concept of satisfiability degrees of formulas in $Ł_n$ which is based on the measure theory on the set Ω_n of all valuations from the set $F(S)$ of all well-formed formulas into L_n. In Section 3 we introduce the concept of consistency degrees of theories after taking on a deep analysis on the inconsistency of theories. This idea has a good intuitive meaning and it can be easily generalized by replacing the contradiction $\bar{0}$ with a general formula A to establish a method of graded reasoning in Ł$_n$. At the same time we propose several other methods of graded reasoning of which comparison is given. All these issues will be discussed in Section 4. In Section 5, we propose syntactically as well as semantically a new complete method of graded reasoning in Ł$_n$.

2 Preliminaries

First let us recall the representation of Łukasiewicz propositional n-valued logic system Ł$_n$.

Following the ideas of [2,4,12], for the syntactic considerations we assume that the formalized language of Ł$_n$ has to include the following items:

i. an (countably) infinite set $S=\{p_1,p_2,\cdots\}$ of propositional variables,
ii. the logical connectives of Ł$_n$: $\neg$(read "not") and $\rightarrow$ (read "Łukasiewicz implication"),
iii. the brackets "(","")" and the comma "," as punctuation symbols which support the unique readability of the (well-formed) formulas.

The set $F(S)$ of well-formed formulas is generated in the following way:

i. each propositional variable $p_i \in S$ (called an atomic formula) is in $F(S)$,
ii. if A and B are in $F(S)$ then so are $\neg A$ and $A \rightarrow B$.

The set of truth degrees of Ł$_n$ here is

$$L_n = \{0, \frac{1}{n-1}, \cdots, \frac{n-2}{n-1}, 1\},$$

and L_n has two operations: $\neg$(negation) and $\rightarrow$(Łukasiewicz implication operator), which are defined as follows respectively,

$$\neg x = 1 - x,$$
$$x \rightarrow y = (1-x+y)\wedge 1, \quad x,y \in L_n.$$

Giving semantic interpretation to a formula $A \in F(S)$ means that we associate a value of truth $v(A) \in L_n$ to A, in other words, we define valuation functions $v: F(S) \rightarrow L_n$. Each valuation $v: F(S) \rightarrow L_n$ which assigns to each well-formed formula A a truth degree $v(A)$ is a homomorphism, i.e,., $v(\neg A) = \neg v(A) = 1 - v(A), v(A \rightarrow B) = v(A) \rightarrow v(B) = (1 - v(A) + v(B)) \wedge 1, A, B \in F(S)$. The set of all valuations is denoted by Ω_n.

A formula A is called a tautology if $v(A)=1$ for each valuation v; A is called a contradiction if $\neg A$ is a tautology; A and B are called logically equivalent if $v(A)=v(B)$ for each valuation v.

Thus a tautology is a formula that is absolutely true under any valuation. Indeed, we can choose some formulas that are tautologies for the axioms and develop a logical calculus such that the provable formulas are just the tautologies of Ł$_n$. In other words, Ł$_n$ can be axiomatizable [2,13]. Note that since Ł$_2$ is just classical two-valued logic system C_2, we assume in the following theorem $n \geq 3$.

Theorem 2.1 (Axiomatizability Theorem for Ł$_n$). (Gottwald [2]). For each integer $n \geq 3$ an axiomatization of the n-valued Łukasiewicz propositional logic system Ł$_n$ is given by the axiom schemata

(L_n1) $A \to (B \to A)$,
(L_n2) $(A \to B) \to ((B \to C) \to (A \to C))$,
(L_n3) $(\neg A \to \neg B) \to (B \to A)$,
(L_n4) $((A \to B) \to B) \to ((B \to A) \to A)$,
(L_n5) $n \cdot A \to (n-1) \cdot A$,
(L_n6) $(n-1) \cdot (A^k \underline{\vee} (\neg A \& (k-1) \cdot A))$ for each $1 < k < n-1$ for which $k-1$ does not divide $n-1$
together with the rule of detachment (MP) w.r.t. the Łukasiewicz implication $\to$ as an inference rule, where

$$A \& B = \neg(A \to \neg B),$$
$$A \underline{\vee} B = \neg A \to B,$$
$$A^1 = A, A^k = A^{k-1} \& A,\ k = 2, 3, \cdots,$$
$$1 \cdot A = A, m \cdot A = ((m-1) \cdot A) \underline{\vee} A,\ m = 2, 3, \cdots.$$

A theory over $Ł_n$ is a set of formulas. A deduction of $A \in F(S)$ from a theory Γ, in symbols, $\Gamma \vdash A$ (or, more precisely, $\Gamma \vdash_n A$), is a sequence $A_1, A_2, \cdots, A_n = A$ of formulas whose each member is either an axiom of $Ł_n$ or a member of Γ or follows from some preceding members of the sequence using the deduction rule MP. In this case, we say that A is a consequence of Γ. The set of all consequences of Γ is denoted by $D(\Gamma)$. By a proof of A we shall henceforth mean a deduction of A from the empty set. We shall also write $\vdash_n A$ in place of $\emptyset \vdash_n A$ and call A a theorem of $Ł_n$. A theory Γ is called inconsistent if $\Gamma \vdash_n \bar{0}$, otherwise consistent, where $\bar{0}$ is a refutable formula, i.e., $\vdash_n \neg\bar{0}$ holds.

As has been mentioned above, $Ł_n$ is complete [2].

Theorem 2.2. (Gottwald [2]). In $Ł_n, n \geq 2$, A is a theorem iff A is a tautology.

Considering the length and concision of the present paper, we do not intend to introduce the continuous-valued Łukasiewicz propositional fuzzy logic Ł in detail. The formal language of Ł is the same as that of $Ł_n (n \geq 3)$, and Ł takes $(L_n1) - (L_n4)$ as its axiom schemata and the MV-unit interval $([0,1], \neg, \to)$ as its set of truth degrees, where $\neg x = 1 - x, x \to y = (1 - x + y) \wedge 1, x, y \in [0,1]$. It is well known that Ł is complete w.r.t. the MV-unit interval $([0,1], \neg, \to)$. Please see [2,4,12].

Deduction theorem is one of the most important theorems in classical logic systems C_2, it says that [6]

$$\Gamma \cup \{A\} \vdash B \ \text{ iff } \ \Gamma \vdash A \to B.$$

Because the necessity part of the deduction theorem above depends on the following axiom of C_2:

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$$

which is not provable in $Ł_n$ or Ł, the deduction theorem is no longer valid in $Ł_n$ in general. Fortunately, it has been proved there exists in Ł a weak version of the deduction theorem (called generalized deduction theorem) [2,4]:

$$\Gamma \cup \{A\} \vdash B \ \text{ iff } \ \exists m \in \mathbf{N}, s.t. \Gamma \vdash A^m \to B.$$

Then a natural question aries: Does there exist in Ł an effective method to determine this parameter m even in some uniform way? The Theorem 3.2 of [15] tells that in Ł there is not one single value of this parameter which applies to all formal theories. However, in $Ł_n$ the parameter m can be decided. There is a semantic version of the generalized deduction theorem which holds true in each of the finitely many-valued systems $Ł_n$ [10]:

$$\Gamma \cup \{A\} \models_n B \ \text{ iff } \ \models_n A^{n-1} \to B,$$

where $\Gamma \models_n B$ means that the class of 1-models $[\Gamma]$ of Γ is a subset of $[B]$ of B, i.e.,

$$[\Gamma] \subseteq [B]$$

and

$$[\Gamma] = \{v \in \Omega_n \mid (\forall A \in \Gamma) v(A) = 1\},$$
$$[B] = \{v \in \Omega_n \mid v(B) = 1\}.$$

Since Ł$_n$ is complete w.r.t. L_n, the semantics and syntax of Ł$_n$ are in perfect harmony and there is no difference between a theorem and a tautology. Whence the syntactic version of the generalized deduction theorem holds obviously in Ł$_n$:

Theorem 2.3. In Ł$_n (n \geq 2)$ holds the generalized deduction theorem:

$$\Gamma \cup \{A\} \vdash_n B \text{ iff } \Gamma \vdash_n A^{n-1} \to B.$$

In what follows, we introduce in Ł$_n$ the concept of satisfiability degrees of formulas proposed by the authors of [18].

Definition 2.1. (Halmos [5]). Suppose that $(X_k, \mathscr{A}_k, \mu_k)(k = 1, 2, \cdots)$ are probability measure spaces, and let $X = \prod_{k=1}^{\infty} X_k$. Then $\prod_{k=1}^{\infty} \mathscr{A}_k$ generates on X a σ-algebra $\mathscr{A}$, and there exists on X a measure μ satisfying the following conditions:

i. $\mathscr{A}$ is the set of consisting of all μ-measurable sets,
ii. For any measurable subset E of $\prod_{k=1}^{m} X_k$, $E \times \prod_{k=m+1}^{\infty} X_k$ is μ-measurable and

$$\mu(E \times \prod_{k=m+1}^{\infty} X_k) = (\mu_1 \times \ldots \times \mu_m)(E).$$

μ is called the infinite product measure of $\mu_1, \mu_2, \cdots$, and $(X, \mathscr{A}, \mu)$ is called the infinite product of $\{(X_k, \mathscr{A}_k, \mu_k)\}_{k=1}^{\infty}$. $(X, \mathscr{A}, \mu)$ is often abbreviated as X if no confusion arises.

Definition 2.2. Suppose that $n \geq 2$ is a fixed natural number, and $(Y, \mathscr{B}, \eta)$ is an evenly distributed probability measure space where $Y = \{y_1, \cdots, y_n\}$, i.e., $\eta(\emptyset) = 0, \eta(Y) = 1$ and $\eta(y_i) = \frac{1}{n}(i = 1, \cdots, n)$. Let $(X_k, \mathscr{A}_k, \mu_k) = (Y, \mathscr{B}, \eta)(k = 1, 2, \cdots)$, and $(X, \mathscr{A}, \mu)$ be the infinite product of $\{(X_k, \mathscr{A}_k, \mu_k)\}_{k=1}^{\infty}$. Then $(X, \mathscr{A}, \mu)$ is called an n-valued logic measure space.

In the following Y will be written as $Y = L_n = \{0, \frac{1}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$, hence we have

$$X_k = L_n = \{0, \frac{1}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}, k = 1, 2, \cdots,$$

and $X = (X, \mathscr{A}, \mu)$ can also be written as L_n^{∞}.

Let $v \in \Omega_n$. Then v is uniquely determined by its restriction $v \mid S$ because $F(S)$ is a free algebra of type $(\neg, \to)$ generated by S. Assume that $v(p_k) = v_k (k = 1, 2, \cdots)$, then an infinite dimensional vector $\overrightarrow{v} = (v_1, v_2, \cdots)$ in L_n^{∞} is obtained. Conversely, let $\overrightarrow{v} = (v_1, v_2, \cdots)$ be any element of L_n^{∞}. Then there exists a unique $v \in \Omega_n$ such that $v(p_k) = v_k (k = 1, 2, \cdots)$. Hence there exists a bijection $\varphi : \Omega_n \to L_n^{\infty}$ defined by $\varphi(v) = \overrightarrow{v}$. In thise sense we say also $\overrightarrow{v}$ a t-model of a formula A if v is a t-model of A, $t \in L_n$ and similarly for a theory Γ.

Definition 2.3. (Wang and Li [18]). Suppose that $A \in F(S)$ in $Ł_n, n \geq 2$, define

$$\tau_n(A) = \sum_{i=0}^{n-1} \frac{i}{n-1} \mu([A]_{\frac{i}{n-1}}) = \sum_{i=1}^{n-1} \frac{i}{n-1} \mu([A]_{\frac{i}{n-1}}),$$

where $[A]_{\frac{i}{n-1}}$ is the class of $\frac{i}{n-1}$-models of A, i.e.,

$$[A]_{\frac{i}{n-1}} = \{\overrightarrow{v} \in L_n^{\infty} \mid \overrightarrow{v} = \varphi(v), v(A) = \frac{i}{n-1}, v \in \Omega_n\}, \quad i = 0, 1, \cdots, n-1.$$

Then $\tau_n(A)$ is called the n-valued satisfiability degree of A in $Ł_n$.

Definition 2.3 is obviously a natural and non-trivial generalization of the case of classical two-valued logic. Please see [16].

The following Proposition is obvious:

Proposition 2.1. Suppose that A and B are formulas in $Ł_n$, $n \geq 2$, then

i. A is a tautology iff $\tau_n(A) = 1$,
ii. A is a contradiction iff $\tau_n(A) = 0$,
iii. $\tau_n(\neg A) = 1 - \tau_n(A)$,
iv. If $\vdash_n A \to B$ holds, then $\tau_n(A) \leq \tau_n(B)$,
v. If A and B are logically equivalent then $\tau_n(A) = \tau_n(B)$.
vi. Define

$$\rho_n(A,B) = 1 - \tau_n((A \to B) \wedge (B \to A)), A, B \in F(S), \tag{1}$$

then it is easy to check that $\rho_n(A,B)$ is a pseudo-metric on $F(S)$.

Example 2.1. i. $\tau_2(p) = \mu([p]_1) = \frac{1}{2}$, and

$$\begin{aligned} \tau_n(p) &= \sum_{i=0}^{n-1} \frac{i}{n-1} \mu([p]_{\frac{i}{n-1}}) \\ &= \sum_{i=0}^{n-1} \frac{i}{n-1} \mu_1(\{\frac{i}{n-1}\}) \\ &= \sum_{i=0}^{n-1} \frac{i}{n-1} \frac{1}{n} \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \frac{i}{n-1} \\ &= \frac{1}{n(n-1)} \sum_{i=0}^{n-1} i \\ &= \frac{1}{n(n-1)} \cdot \frac{n(n-1)}{2} \\ &= \frac{1}{2} \end{aligned} \tag{2}$$

ii.

$$\tau_n(p \to q) = \sum_{i=0}^{n-1} \frac{i}{n-1} \mu([p \to q]_{\frac{i}{n-1}}) \tag{3}$$
$$= \frac{1}{n^2} \sum_{i=1}^{n-1} \frac{i}{n-1} |\overline{p \to q}^{-1}(\frac{i}{n-1})|$$
$$= \frac{1}{n^2(n-1)}(\frac{n(n+1)(n-1)}{2} + \sum_{i=1}^{n-2} i(i+1))$$
$$= \frac{1}{6n^2(n-1)}(5n^2-n)(n-1)$$
$$= \frac{5n-1}{6n},$$

and so $\tau_2(p \to q) = \frac{3}{4}, \tau_3(p \to q) = \frac{7}{9}$.

iii. $\tau_n(p \vee q) = \sum_{i=1}^{n-1} \frac{i}{n-1} \frac{2i+1}{n^2} = \frac{4n+1}{6n}$.

Before giving the limit theorem which builds a bridge between discrete valued Łukasiewicz logic and continuous valued Łukasiewicz logic, we recall the concept of the integrated satisfiability degrees of formulas in Ł [17].

Definition 2.4. (Wang and Leung [17].) Let $A = A(p_1, \cdots, p_m)$ be a formula in $F(S)$ and $\overline{A} = \overline{A}(x_1, \cdots, x_m)$ be the truth degree function of A. Then the integral

$$\tau(A) = \int_{[0,1]^m} \bar{A}(x_1, \cdots, x_m) dx_1 \cdots dx_m$$

is called the integrated satisfiability degree of A in Ł.

Example 2.2. i. $\tau(p) = \int_{[0,1]} \overline{p}(x)dx = \int_{[0,1]xdx} = \frac{1}{2}$.

ii.

$$\begin{aligned} \tau(p \to q) &= \textstyle\int_{[0,1]^2} \overline{p \to q}(x,y)dxdy \\ &= \textstyle\int_{[0,1]^2} x \to ydxdy \\ &= \textstyle\int_{[0,1]^2} (1-x+y) \wedge 1dxdy \\ &= \textstyle\int_{x>y} 1-x+ydxdy + \int_{x \le y} 1dxdy \\ &= \tfrac{1}{3} + \tfrac{1}{2} \\ &= \tfrac{5}{6}. \end{aligned}$$

iii.

$$\begin{aligned} \tau(p \vee q) &= \textstyle\int_{[0,1]^2} \overline{p \vee q}(x,y)dxdy \\ &= \textstyle\int_{[0,1]^2} x \vee ydxdy \\ &= \textstyle\int_0^1 \int_y^1 xdxdy + \int_0^1 \int_x^1 ydydx \\ &= \tfrac{1}{3} + \tfrac{1}{3} \\ &= \tfrac{2}{3}. \end{aligned}$$

Compare Example 2.1 with Example 2.2, we find that $\tau(p) = \lim_{n\to\infty} \tau_n(p) = \frac{1}{2}$, $\tau(p \to q) = \lim_{n\to\infty} \tau_n(p \to q) = \frac{5}{6}$, and $\tau(p \vee q) = \lim_{n\to\infty} \tau_n(p \vee q) = \frac{2}{3}$. In fact, we have the following limit theorem:

Theorem 2.4 (Limit Theorem). (Wang and Li [18]). $\forall A \in F(S)$,

$$\lim_{n\to\infty} \tau_n(A) = \tau(A).$$

3 Consistency degrees of theories based on deduction theorems in Ł$_n$

First let us take an analysis in Ł$_n$ on the inconsistency of a theory. Suppose that Γ is a theory and Γ is inconsistent, then the contradiction $\bar{0}$ is a consequence of Γ, that is to say, $\Gamma \vdash_n \bar{0}$ holds. It follows from the generalized deduction theorem that there exists a finite string of formulas $A_1,\cdots,A_m \in \Gamma$ such that $\vdash_n A_1^{n-1}\&A_2^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}$ holds (note that $A \to (B \to C)$ and $A\&B \to C$ are provably equivalent), i.e., the formula $A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}$ is a tautology. Then the satisfiability degree $\tau_n(A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}) = 1$ following from Proposition 2.1. Conversely, if there is a finite sequence of formulas $A_1,\cdots,A_m \in \Gamma$ such that the satisfiability degree $\tau_n(A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0})$ of $A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}$ is equal to 1, then again following from Proposition 2.1, $A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}$ is a tatutology, equivalently, $\vdash_n A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}$ holds, i.e., $\vdash_n (A_1^{n-1} \to (A_2^{n-1} \to (\cdots(A_m^{n-1} \to \bar{0})\cdots)))$ holds. Hence by the generalized deduction theorem of Ł$_n$ m times, $\Gamma \vdash_n \bar{0}$ is yielded, and so Γ is inconsistent in Ł$_n$. From the above analysis, in order to decide whether a given theory Γ is inconsistent or not, it suffices to calculate the satisfiability degrees $\tau_n(A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0})$ of all possible formulas of the form $A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}$ where $A_1,\cdots,A_m \in \Gamma$. If the satisfiability degree of such a formula is equal to 1, then Γ is inconsistent. However, it may happen during our reasoning that we obtain more satisfiability degrees of such formulas from Γ and it is necessary to decide, which of them should be taken as the result. By the completeness theorem of Ł$_n$, the larger the satisfiability degrees of such formulas are, the closer Γ is to be inconsistent. Therefore, it is natural and reasonable for us using the supremum of satisfiability degrees of all formulas of the form $A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}$, where $A_1,\cdots,A_m \in \Gamma$, to measure the inconsistency of Γ.

Definition 3.1. Suppose that Γ is a theory of Ł$_n, n \geq 2, 2^{(\Gamma)}$ is the set of all finite subsets of $\Gamma, \Sigma = \{A_1,\cdots,A_m\} \in 2^{(\Gamma)}$. Let

$$\Sigma^n \to \bar{0} = \begin{cases} A_1^{n-1}\&\cdots\&A_m^{n-1} \to \bar{0}, & m > 0, \\ \bar{0}, & m = 0, \end{cases}$$

and define

$$\mu_n(\Gamma) = \sup\{\tau_n(\Sigma^n \to \bar{0}) \mid \Sigma \in 2^{(\Gamma)}\}.$$

Then $\mu_n(\Gamma)$ is called the degree of entailment of $\bar{0}$ from Γ, or say, $\bar{0}$ is a consequence of Γ in the degree $\mu_n(\Gamma)$.

Remark 3.1.
i. It is easy to verify that $\forall A, B \in F(S), A\&B \to C$ and $B\&A \to C$ are logically equivalent, hence the definition of $\Sigma^n \to \bar{0}$ does not depend on the order of $A_i's$,
ii. Since $\tau_n(A\&B \to C) \geq \tau_n(A \to C) \wedge \tau_n(B \to C)$, we have $\tau_n(\Sigma_1^n \to \bar{0}) \leq \tau_n(\Sigma_2^n \to \bar{0})$, where $\Sigma_1 \subseteq \Sigma_2, \Sigma_1, \Sigma_2 \in 2^{(\Gamma)}$,
iii. Suppose that Γ is finite, then it follows from (ii) that $\mu_n(\Gamma) = \tau_n(\Gamma^n \to \bar{0})$,
iv. If one introduces in Ł$_n$ the concept of divergence degrees of theories by means of the satisfiability degrees of formulas as done in [17]:

$$div_n(\Gamma) = \sup\{\rho_n(A,B) \mid A, B \in D(\Gamma)\},$$

where ρ_n is defined by Proposition 2.1 (vi). Then it can be proved in Ł$_n$ that $div_n(\Gamma) = \mu_n(\Gamma)$ (Please see the proof of Theorem 3.8 of [22]). But in the present paper we prefer $\mu(\Gamma)$ because it not only has a very strong intuitive meaning-the extent to which Γ is inconsistent, but also can be easily generalized to give a method of graded reasoning in Ł$_n$.

The calculation of $\mu_n(\Gamma)$ can be simplified as follows:

Theorem 3.1. Suppose that Γ is a theory of Ł$_n$, $n \geq 2$, then

$$\mu_n(\Gamma) = 1 - \inf\{\tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1}) \mid A_1, \cdots, A_m \in \Gamma, m \in \mathbf{N}\}.$$

Proof. It follows from the fact $\neg A$ and $A \to \bar{0}$ are logically equivalent that

$$\begin{aligned}\mu_n(\Gamma) &= \sup\{\tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1} \to \bar{0}) \mid A_1, \cdots, A_m \in \Gamma, m \in \mathbf{N}\} \\ &= \sup\{\tau_n(\neg(A_1^{n-1} \& \cdots \& A_m^{n-1})) \mid A_1, \cdots, A_m \in \Gamma, m \in \mathbf{N}\} \\ &= \sup\{1 - \tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1}) \mid A_1, \cdots, A_m \in \Gamma, m \in \mathbf{N}\} \\ &= 1 - \inf\{\tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1}) \mid A_1, \cdots, A_m \in \Gamma, m \in \mathbf{N}\}.\end{aligned}$$

The proof is completed.

Corollary 3.1. Suppose that $\Gamma = \{A_1, \cdots, A_m\}$ is a finite theory of Ł$_n$, then

$$\mu_n(\Gamma) = 1 - \tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1}).$$

Example 3.1. Calculate $\mu_n(\Gamma)$ for (i) $\Gamma = \emptyset$, (ii) $\Gamma = \{p\}$, where p is a propositional variable, (iii) $\Gamma = S = \{p_1, p_2, \cdots\}$.

Solution 3.1. 1. If $\Gamma = \emptyset$, then $\forall \Sigma \in 2^{(\Gamma)}, \Sigma = \emptyset, \Sigma^n \to \bar{0} = \bar{0}$, and so $\mu_n(\Sigma^n \to \bar{0}) = 0$. Hence $\mu_n(\Gamma) = 0$.

2. Let $\Gamma = \{p\}$, then it follows from Corollary 3.1 that

$$\mu_n(\Gamma) = 1 - \tau_n(p^{n-1}).$$

Recall the Łukasiewicz t-norm $* : [0,1]^2 \to [0,1]$ [7] :

$$x * y = (x + y - 1) \vee 0$$

and it is easy to verify that $x^m = (mx - (m-1)) \vee 0, x \in [0,1]$ [7]. $\forall x \in L_n$, if $x \leq \frac{n-2}{n-1}$, then $(n-1)x - (n-2) \leq 0$, so $x^{n-1} = 0$, and $x^{n-1} = 1$ iff $x = 1$. Thus $\mu([p^{n-1}]_t) = 0$ if $t \leq \frac{n-2}{n-1}$ and $\mu([p^{n-1}]_1) = \frac{1}{n}$. Therefore $\tau_n(p^{n-1}) = \frac{1}{n}$, and

$$\mu_n(\Gamma) = 1 - \frac{1}{n}.$$

3. Since $\vdash (p_1^{n-1} \& \cdots \& p_m^{n-1}) \to (p_1 \wedge \cdots \wedge p_m)$ holds in Ł, so holds it in the n-valued extension Ł$_n$ of Ł. It is left to the reader to check that

$$\tau_n(p_1 \wedge \cdots \wedge p_m) = \frac{1}{n^m(n-1)} \sum_{k=1}^{n-1} k^m \to 0 (m \to \infty).$$

Therefore

$$\begin{aligned}\mu_n(S) &= 1 - \inf\{\tau_n(p_1^{n-1} \& \cdots \& p_m^{n-1}) \mid p_1, \cdots, p_m \in S, m \in \mathbf{N}\} \\ &\geq 1 - \inf\{\tau_n(p_1 \wedge \cdots \wedge p_m) \mid p_1, \cdots, p_m \in S, m \in N\} \\ &= 1 - \inf\{\tfrac{1}{n^m(n-1)} \sum_{k=1}^{n-1} k^m \mid m \in \mathbf{N}\} \\ &= 1\end{aligned}$$

and so $\mu_n(S) = 1$.

But $\Gamma = S$ is not inconsistent in $Ł_n, n \geq 2$. In fact, suppose on the contrary that Γ is inconsistent, then $\Gamma \vdash_n \bar{0}$ holds. Hence it follows from the generalized deduction theorem of $Ł_n$ that there exist $p_1, \cdots, p_m \in \Gamma$ such that $\vdash_n p_1^{n-1} \& \cdots \& p_m^{n-1} \to \bar{0}$ holds. Choose $v \in \Omega_n$ satisfying $v(p_1) = \cdots = v(p_m) = 1$, then $v(p_1^{n-1} \& \cdots \& p_m^{n-1}) = v(p_1^{n-1}) * \cdots * v(p_m^{n-1}) = 1$, where $*$ is the Łukasiewicz t-norm. Thus $v(p_1^{n-1} \& \cdots \& p_m^{n-1} \to \bar{0}) = 1 \to 0 = 0$, contradicting the assumption that $p_1^{n-1} \& \cdots \& p_m^{n-1} \to \bar{0}$ is a theorem. The proof is completed.

Theorem 3.2. Let Γ be a theory of $Ł_n, n \geq 2$. If Γ is inconsistent then $\mu_n(\Gamma) = 1$, but not vice visa.

Proof. The proof is straightforward and so it is omitted. For counterexample, please see Example 2.2(iii).

From the analysis at the beginning of this section, one sees that $\mu_n(\Gamma)$ is an ideal index to measure the extent of the inconsistency of Γ. Perhaps this hints us the idea that one may define the consistency degree $consist_n(\Gamma)$ of Γ to be $1 - \mu_n(\Gamma)$, but this idea has a shortcoming that it could not distinguish theories with $\mu_n(\Gamma) = 1$ from inconsistent theories as shown in Example 2.2 and Theorem 2.4. Hence one has to revise the seemingly reasonable definition of $consist_n(\Gamma) = 1 - \mu_n(\Gamma)$.

Definition 3.2. Suppose that Γ is a theory of $Ł_n, n \geq 2$, define

$$i_n(\Gamma) = \max\{[\tau_n(\Sigma^n \to \bar{0})] \mid \Sigma \in 2^{[\Gamma]}\},$$

and $i_n(\Gamma)$ is called the polar index of Γ in $Ł_n$, where $\Sigma^n \to \bar{0}$ is defined as in Definition 3.1.

Theorem 3.3. Suppose that Γ is a theory of $Ł_n, n \geq 2$, then

1. Γ is consistent iff $i_n(\Gamma) = 0$,
2. Γ is inconsistent iff $i_n(\Gamma) = 1$.

Proof. Since the concept of consistency of a theory is crisp rather than fuzzy, and $i_n(\Gamma) \in \{0, 1\}$, it suffices to prove (ii). Suppose that Γ is inconsistent, i.e., $\Gamma \vdash \bar{0}$, then by the generalized deduction theorem of $Ł_n$, there exist $A_1, \cdots, A_m \in \Gamma$ such that $A_1^{n-1} \& \cdots \& A_m^{n-1} \to \bar{0}$ is a theorem, and it follows from the completeness theorem of $Ł_n$ and Proposition 2.7 that $\tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1} \to \bar{0}) = 1$. So $[\tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1})] = 1$ and $i_n(\Gamma) = 1$. Conversely, if $i_n(\Gamma) = 1$, then there exist $A_1, \cdots, A_m \in \Gamma$ such that $\tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1} \to \bar{0}) = 1$. Again by completeness theorem of $Ł_n$ and Proposition 2.7, $\vdash_n A_1^{n-1} \& \cdots \& A_m^{n-1} \to \bar{0}$ holds. Therefore $\Gamma \vdash_n \bar{0}$ holds by the generalized deduction theorem of $Ł_n$, and hence Γ is inconsistent.

Definition 3.3. Suppose that Γ is a theory of $Ł_n, n \geq 2$, define

$$consist_n(\Gamma) = 1 - \frac{1}{2}\mu_n(\Gamma)(1 + i_n(\Gamma))$$

and call $consist_n(\Gamma)$ the consistency degree of Γ in $Ł_n$.

Theorem 3.4. Suppose that Γ is a theory of $Ł_n, n \geq 2$, then

1. Γ is completely consistent, i.e., all members of $D(\Gamma)$ are tautologies, iff $consist_n(\Gamma) = 1$,
2. Γ is consistent iff $\frac{1}{2} \leq consist_n(\Gamma) \leq 1$,
3. Γ is consistent and $\mu_n(\Gamma) = 1$ iff $consist_n(\Gamma) = \frac{1}{2}$,
4. Γ is inconsistent iff $consist_n(\Gamma) = 0$.

Proof. 1. It is easy to verify that Γ is completely consistent iff $\mu_n(\Gamma) = 0$, hence (i) holds

2. On account of Theorem 3.3(i) Γ is consistent iff $i_n(\Gamma) = 0$, and this is equivalent to $\frac{1}{2} \leq consist_n(\Gamma) = 1 - \frac{1}{2}\mu_n(\Gamma) \leq 1$.
3. It follows directly from Theorems 3.2 and 3.3.
4. Let Γ be inconsistent. Then $\mu_n(\Gamma) = 1$ and $i_n(\Gamma) = 1$ by Theorems 3.2 and 3.3 respectively. Hence $consist_n(\Gamma) = 1 - \frac{1}{2} \times 1 \times (1+1) = 0$. Conversely, if Γ is consistent, then $\frac{1}{2} \leq consist_n(\Gamma) \leq 1$, a contradiction.

Remark 3.2. Corresponding to the Limit Theorem (Theorem 2.4) on satisfiability degrees of formulas, here are similar limit theorems for $\mu_n(\Gamma), i_n(\Gamma)$ and $consist_n(\Gamma)$ respectively:

$$\lim_{n\to\infty} \mu_n(\Gamma) = \mu(\Gamma),$$
$$\lim_{n\to\infty} i_n(\Gamma) = i(\Gamma),$$
$$\lim_{n\to\infty} consist_n(\Gamma) = consist(\Gamma).$$

As for the definitions of $\mu(\Gamma), i(\Gamma)$ and $consist(\Gamma)$ we refer to [22].

4 Methods of graded reasoning in Ł$_n$

As has been pointed out in Remark 3.1(iv), Definition 3.1 indeed offers a method to evaluate the extent to which the contradiction $\bar{0}$ is a consequence of a theory Γ.

If we replace $\bar{0}$ by a general formula A, then the corresponding analysis at the beginning of Section 3 also holds in Ł$_n$. Hence we get a method to measure the extent to which a formula A is a consequence of a theory Γ.

Definition 4.1. Suppose that Γ is a theory of Ł$_n, n \geq 2, 2^{(\Gamma)}$ is the set of all finite subsets of $\Gamma, \Sigma = \{A_1, \cdots, A_m\} \in 2^{(\Gamma)}$, and $A \in F(S)$ is a well-formed formula. Let

$$\Sigma^n \to A = \begin{cases} A_1^{n-1} \& \cdots \& A_m^{n-1} \to A, & m > 0, \\ A, & m = 0, \end{cases}$$

and define

$$\mu_n(A,\Gamma) = \sup\{\tau_n(\Sigma^n \to A) \mid \Sigma \in 2^{(\Gamma)}\}.$$

Then $\mu_n(A,\Gamma)$ is called the degree of entailment of A from Γ, or say A is a consequence of Γ in the degree $\mu_n(A,\Gamma)$.

Note that Remark 3.1 holds also when $\bar{0}$ is replaced by A, and obviously $\mu_n(\bar{0},\Gamma) = \mu_n(\Gamma)$.

Similar to theorem 3.2, we have the following theorem.

Theorem 4.1. Let A be not a theorem and Γ be a theory of Ł$_n$. If A is a consequence of Γ then $\mu_n(A,\Gamma) = 1$, but not vice versa in general.

Clearly Theorem 3.2 is a special case of Theorem 4.1. It is not difficult to verify that $\sup\{\tau_n(\Sigma^n \to A \mid \Sigma \in 2^{(\Gamma)})\} \geq \sup\{\tau_n(p_2 \wedge \cdots \wedge p_m \to p_1) \mid m \in \mathbf{N}, m \geq 2\} = 1$, where $\Gamma = \{p_2 \wedge \cdots \wedge p_m \mid m = 2, 3, \cdots\}$ and $A = p_1$, and it is trivial to show that $A \notin D(\Gamma)$. Theorem 4.1 tells us that $\mu_n(A,\Gamma) = 1$ does not mean that A is (100%) a consequence of Γ.

Definition 4.2. Suppose that Γ is a theory of $Ł_n, n \geq 2, A \in F(S)$, define

$$i_n(A,\Gamma) = \max\{[\Sigma^n \to A] \mid \Sigma \in 2^\Gamma\},$$

and call $i_n(A,\Gamma)$ the polar index of Γ w.r.t. A in $Ł_n$.

Theorem 4.2. Suppose that Γ is a theory of $Ł_n, n \geq 2$, then

1. $A \in D(\Gamma)$ iff $i_n(A,\Gamma) = 1$,
2. $A \notin D(\Gamma)$ iff $i_n(A,\Gamma) = 0$.

Proof. The proof is trivial.

Definition 4.3. Let Γ be a theory of $Ł_n, n \geq 2, A \in F(S)$, define

$$\mu_n^*(A,\Gamma) = \frac{1}{2}\mu_n(A,\Gamma)(1 + i_n(A,\Gamma)),$$

and call $\mu_n^*(A,\Gamma)$ also the degree of entailment of A from Γ, or say also A is a consequence of Γ in the degree $\mu_n^*(A,\Gamma)$.

Theorem 4.3. Suppose that Γ is a theory of $Ł_n, n \geq 2, A \in F(S)$, then

1. $A \in D(\Gamma)$ iff $\mu_n^*(A,\Gamma) = 1$,
2. $A \notin D(\Gamma)$ iff $\frac{1}{2}\tau_n(A) \leq \mu_n^*(A,\Gamma) \leq \frac{1}{2}$,
3. $A \notin D(\Gamma)$ and $\mu_n(A,\Gamma) = 1$ iff $\mu_n^*(A,\Gamma) = \frac{1}{2}$.

Proof. The proof is analogous to that of Theorem 3.4 and so it is omitted.

In the following we give another two methods of graded reasoning in $Ł_n$ by measuring the distance of A to $D(\Gamma)$.

Definition 4.4. Let Γ be a theory of $Ł_n, n \geq 2, A \in F(S)$, put

$$Con_{\rho_n}(A,\Gamma) = 1 - \rho_n(A, D(\Gamma)) = 1 - \inf\{\rho_n(A,B) \mid B \in D(\Gamma)\},$$

and call A a ρ_n-consequence of Γ in the degree $Con_{\rho_n}(A,\Gamma)$.

The following theorem reveals the relationship between ρ_n and μ_n.

Theorem 4.4. Let Γ be a theory of $Ł_n, n \geq 2, A \in F(S)$, then

$$Con_{\rho_n}(A,\Gamma) = \mu_n(A,\Gamma).$$

Proof. On one hand,

$$\begin{aligned}
\rho_n(A, D(\Gamma)) &= \inf\{\rho_n(A,B) \mid B \in D(\Gamma)\} \\
&= \inf\{1 - \tau_n((A \to B) \wedge (B \to A)) \mid B \in D(\Gamma)\} \\
&= 1 - \sup\{\tau_n((A \to B) \wedge (B \to A)) \mid B \in D(\Gamma)\} \\
&\geq 1 - \sup\{\tau_n(B \to A) \mid B \in D(\Gamma)\} \\
&= 1 - \sup\{\tau_n(A_1^{n-1} \& \cdots \& A_m^{n-1} \to A) \mid A_1, \cdots, A_m \in \Gamma, m \in \mathbf{N}\} \\
&= 1 - \mu_n(A,\Gamma).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mu_n(A,\Gamma) &= \sup\{\tau_n(A_1^{n-1}\&\cdots\&A_m^{n-1}\to A)\mid A_1,\cdots,A_m\in\Gamma, m\in\mathbf{N}\}\\ &= \sup\{\tau_n((A_1^{n-1}\&\cdots\&A_m^{n-1}\to A)\wedge(A\to A))\mid A_1,\cdots,A_m\in\Gamma, m\in\mathbf{N}\}\\ &= \sup\{\tau_n((A_1^{n-1}\&\cdots\&A_m^{n-1})\vee A)\to A)\mid A_1,\cdots,A_m\in\Gamma, m\in\mathbf{N}\}\\ &= \sup\{1-\rho_n(A,(A_1^{n-1}\&\cdots\&A_m^{n-1})\vee A)\mid A_1,\cdots,A_m\in\Gamma, m\in\mathbf{N}\}\\ &= 1-\inf\{\rho_n(A,(A_1^{n-1}\&\cdots\&A_m^{n-1})\vee A)\mid A_1,\cdots,A_m\in\Gamma, m\in\mathbf{N}\}\\ &\le 1-\rho_n(A,D(\Gamma)).\end{aligned}$$

Note that the last equality hods because $(A_1^{n-1}\&\cdots\&A_m^{n-1})\vee A\in D(\Gamma)$ whenever $A_1,\cdots,A_m\in\Gamma$. Hence the proof is completed.

Corollary 4.1. Suppose that Γ is a theory of $Ł_n, n\ge 2, A\in F(S)$, then $A\in\partial(D(\Gamma))$, i.e., $\rho_n(A,D(\Gamma)) = 0$ but $A\notin D(\Gamma)$, iff $\mu_n^*(A,\Gamma)=\frac{1}{2}$.

Proof. It follows directly from Theorem 4.4, Definition 4.3 and Theorem 4.3.

In the following we give another method of graded reasoning in $Ł_n$ where we need the concept of Hausdorff distance [11].

Definition 4.5. Let (X,d) be a pseudo-metric space, and let U,V be non-empty subsets of X. Define

$$\begin{aligned}&d(x,U)=\inf\{d(x,u)\mid u\in U\},\\ &H^*(U,V)=\sup\{d(u,V)\mid u\in U\},\\ &H(U,V)=\max\{H^*(U,V),H^*(V,U)\}.\end{aligned}$$

Then H is a pseudo-metric on $\mathscr{P}(X)-\{\emptyset\}$ which is called the Hausdorff distance induced by d.

Definition 4.6. Suppose that Γ is a theory of $Ł_n$, A is a formula in $F(S)$. Define

$$Con_H(A,\Gamma)=1-\inf\{H(D(\Gamma),D(\Sigma))\mid \Sigma \text{ is a theory such that} \Sigma\vdash A\}$$

where H is the Hausdorff distance induced by ρ_n on $F(S)$, then we say that A is an H-consequence of Γ in the degree $Con_H(A,\Gamma)$.

Theorem 4.5. Suppose that Γ is a theory over $Ł_n$, A is a formula, then

$$Con_H(A,\Gamma)\le\mu_n(A,\Gamma)=Con_{\rho_n}(A,\Gamma).$$

Proof. It suffices to show that for every $a\in[0,1]$, $\rho_n(A,\Gamma)<a$ whenever $\inf\{H(D(\Gamma),D(\Sigma))\mid\Sigma$ is a theory such that$\Sigma\vdash A\}<a$. Suppose that $\inf\{H(D(\Gamma),D(\Sigma))\mid\Sigma$ is a theory such that$\Sigma\vdash A\}<a$, then there exists a theory Σ such that

$$H(D(\Gamma),D(\Sigma))<a \text{ and } \Sigma\vdash A.$$

Since $A\in D(\Gamma)$, we have

$$\rho_n(A,D(\Gamma))\le H^*(D(\Sigma),D(\Gamma))\le H(D(\Gamma),D(\Sigma))<a.$$

This shows that $Con_H(A,\Gamma)\le Con_{\rho_n}(A,\Gamma)$. The proof is completed.

5 A new complete method of graded reasoning in $Ł_n$

In this section we extend the idea of [20] to $Ł_n$ and propose syntactically as well as semantically a new method of graded reasoning based on the pseudo-metric ρ_n. We prove the completeness theorem.

In the following we say that a sequence of formulas $\omega = \omega_1 \cdots \omega_n$ is inductive if there is a deduction of ω_n from its preceding members $\omega_1, \cdots, \omega_{n-1}$. The pseudo-metric ρ_n is abbreviated as ρ. For every theory Γ, let $\Gamma^* = \Gamma \cup \mathscr{A}$ where $\mathscr{A}$ is the set of axioms of $Ł_n$.

Definition 5.1. Let Γ be a theory over $Ł_n$, and $\omega = \omega_1 \cdots \omega_n$ a sequence of formulas. Define the deviation degree $\rho(\Gamma, \omega)$ to which ω is a deduction from Γ as follows:

1. If $n = 1$ then $\rho(\Gamma, \omega) = \rho(\omega, \Gamma^*)$,
2. If $n > 1$, then

$$\rho(\Gamma, \omega) = \rho(\omega_n, \Gamma^*) \wedge (\vee\{\rho(\omega_i, \Gamma^*) \vee \rho(\omega_i, \Gamma^*) \mid \{\omega_i, \omega_j\} \vdash \omega_n\}).$$

Definition 5.2. Let Γ be a theory over $Ł_n$, and A a formula. Then the derivation degree $Ded(A, \Gamma)$ to which Γ syntactically implies A is defined by

$$Ded(A, \Gamma) = 1 - \wedge\{\rho(\Gamma, \omega) \mid \omega \text{ is an inductive sequence, } \omega_n = A\}.$$

Remark 5.1. Suppose that $\Gamma \vdash A$, then it can be proved inductively from Definitions 5.1 and 5.2 that $Ded(A, \Gamma) = 1$. Hence Definition 5.2 is a generalization of the fact that $\Gamma \vdash A$.

Definition 5.3. Let Γ be a theory over $Ł_n$ and A a formula. Then the derivation degree $Con(A, \Gamma)$ to which Γ semantically implies A is defied by

$$Con(A, \Gamma) = 1 - \wedge\{\rho(\Sigma, \Gamma^*) \mid \Sigma \text{ is a finite theory }, \Sigma \models A\},$$

where

$$\rho(\Sigma, \Gamma^*) = \vee\{\rho(A, \Gamma^*) \mid A \in \Sigma\}.$$

Remark 5.2. Similar to Remark 5.1, we can see that Definition 5.3 is indeed a generalization of the concept $\Gamma \models A$. In fact, suppose that $\Gamma \models A$, then it follows from the standard completeness theorem of $Ł_n$ that $\Gamma \vdash A$, hence Γ has a finite subset Σ such that $\Sigma \vdash A$ and therefore again by the standard completeness theorem of $Ł_n$ $\Sigma \models A$. It follows form $\Sigma \subseteq \Gamma$ that $\rho(\Sigma, \Gamma^*) = 0$ and hence $Con(A, \Gamma) = 1$.

Theorem 5.1 (Complete theorem of graded reasoning). Let Γ be a theory over $Ł_n$ and A a formula. Then

$$Ded(A, \Gamma) = Con(A, \Gamma).$$

Proof. We show now that

$$1 - Ded(A, \Gamma) = 1 - Con(A, \Gamma).$$

For the "$\leq$" part, suppose that $1 - Con(A, \Gamma) < \varepsilon (\varepsilon > 0)$, then there is a finite theory Σ satisfying

$$\rho(\Sigma, \Gamma^*) < \varepsilon \text{ and } \Sigma \models A.$$

It follows from the standard completeness theorem of $Ł_n$ (Theorem 2.2) that $\Sigma \vdash A$. Let $\Sigma = \{\omega_1, \cdots, \omega_{n-1}\}$, then $\omega = \omega_1 \cdots \omega_{n-1} A$ is an inductive sequence of formulas. It is easy to show by the induction on the length of ω that $\rho(\Gamma, \omega) < \varepsilon$. Hence $1 - Ded(A, \Gamma) < \varepsilon$.

For the "$\geq$" part, suppose that $1 - Ded(A, \Gamma) < \varepsilon (\varepsilon > 0)$, then there is an inductive sequence $\omega = \omega_1 \cdots \omega_n$ such that

$$\rho(\Gamma, \omega) < \varepsilon \text{ and } \omega_n = A.$$

It follows from Definition 5.1 that at least one of the following items (i) and (ii) holds:

1. $\rho(A, \Gamma^*) = \rho(\omega_n, \Gamma^*) < \varepsilon$. In this case let $\Sigma = \{A\}$, then Σ is a finite theory and $\Sigma \models A$ holds. It follows that $\rho(\Sigma, \Gamma^*) = \rho(A, \Gamma^*) < \varepsilon$, and so $1 - Con(A, \Gamma) < \varepsilon$.
2. There exist $i, j < n$ such that $\{\omega_i, \omega_j\} \vdash A$ (hence by Theorem 2.2 $\{\omega_i, \omega_j\} \models A$) and

$$\rho(\omega_i, \Gamma^*) < \varepsilon, \rho(\omega_j, \Gamma^*) < \varepsilon.$$

   In this case let $\Sigma = \{\omega_i, \omega_j\}$, then $\Sigma \models A$. Moreover, $\rho(\Sigma, \Gamma^*) = \rho(\omega_i, \Gamma^*) \vee \rho(\omega_j, \Gamma^*) < \varepsilon$. Thus we have $1 - Con(A, \Gamma) < \varepsilon$. The proof is completed.

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Economic Applications of Fuzzy Subset Theory and Fuzzy Logic: A Brief Survey

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Abstract. The direction of modern innovation is to impart human-like intelligence to models and machines in order to aid in decision-making. To achieve this goal: the philosophy of vagueness is of central importance. The mathematics of uncertainty attempts to take these principles and create a structure with which people can build such models. One of these methods is the theory of fuzzy subsets and fuzzy logic.
This survey paper looks at some of the current applications of fuzzy subset theory and fuzzy logic in economics from decision-making in policy to microeconomic theory. Fuzzy logic is a great example for mathematics of uncertainty and its successful applications to economics raise a great deal of hope that all social sciences and biological sciences can benefit from this new paradigm shift. However, this is merely a short survey and only begins to address this important academic debate.

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1 Introduction: Vagueness

"This, of course, is the answer to the old puzzle about the man who went bald. It is supposed that at first he was not bald, that he lost his hairs one by one, and that in the end he was bald; therefore, it is argued, there must have been one hair the loss of which converted him into a bald man. This, of course, is absurd. Baldness is a vague conception; some men are certainly bald, some are certainly not bald, while between them there are men of whom it is not true to say they must be either be bald or not bald. The law of excluded middle is true when precise symbols are employed, but it is not true when symbols are vague, as, in fact, all symbols are."

Bertrand Russell
Vagueness, 1923

"No man means all he says, and yet very few say all they mean,
for words are slippery and thought is viscous."

Henry Brooks Adams,
The Education of Henry Adams, 1907

Traditional mathematics is the language of precision. Statements are either true or false, black or white, right or wrong. In its pure state, the concept of gray does not exist. It is only introduced when the precise model is used to mimic or explain an inherently imprecise state of reality. Human language, information-processing, and decision-making are all based on approximations and "vague" notions, the way the human mind actually processes information. How does a language based on precision attempt to explain, much less, to model a language that is essentially imprecise?

For example, the main challenge of economics is to model human interaction. In order to model such interactions, economics has traditionally used both linguistic and mathematical models to approximate human rationality or thought to draw a useful conclusion for decision-making. Though the area of society in question may change, the aim is the same; and in order to model human interaction, some assumptions about people in general must be made.

The central assumption in classical economics is rationality. Simply stated, traditional economic models assume that people will always make optimal choices to maximize their perceived utility. However, utility is supposed to represent an aggregation of all of an individual's preferences. Hence, utility is a symbol which, according to Russell, is a vague concept, and it stands to reason that since the basic premise of economic theory is based on a vague notion, traditional mathematical will frequently fail to adequately model a given situation. Empirically, this seems to be the case. It also stands to reason that if we are to improve upon existing models, that system that can adequately deal with vagueness be used. Here we find statistics and probability theory's popularity; however, such tools only model certain aspects of the uncertainty or likelihood of an event triggered by random generation. These methods still require precise variables and do not offer solutions to many other uncertain situations. It must be coupled with yet another model and a person's reasoned application of the output to the other model.

Given this gap between methodology and actuality, economists have been strangely reluctant to harness the power of modeling with mathematics of uncertainty, which is actually concerned with using the vagueness of concepts (such as "high" confidence levels) and constructing solutions based on them, not simply modeling the vagueness itself [10] . This decision-making power is the core of practical economic theory, and herein lays the growing importance of the mathematics of uncertainty.

The mathematics of uncertainty contains many theories and methodologies; however, the focus of this paper will simply be the theory of fuzzy subsets and fuzzy logic as well as its current applications in economics. This paper will have two main sections. The first portion will be an overview of fuzzy subset theory and fuzzy logic. The second section will hope to demonstrate some ways in which fuzzy theory has been applied to both theoretical and practical economics. The initial section will deal with applications in theoretical economics, dealing with topics such as game theory, oligopoly theory, and utility modeling. The second portion will focus on practical applications, particularly in policy. These will include topics like forecasting, international policy, and financial analysis[3]. This paper will by no means pretend to be a complete overview of the subject but simply a snap shot of some essential research that has been done. The final portion will simply be this student's view on the future of fuzzy theory in economics and the importance of its continued study and application in this area.

The Theory of Fuzzy Subsets: A Brief Overview

[3] Applications in operations and production management is intentionally unrepresented as this paper hopes to show less-known applications of fuzzy logic to economic theory

"The theory of fuzzy subsets is, in effect,
a step toward a rapprochement
between the precision of classical mathematics
and the pervasive imprecision of the real world.."

L. A. Zadeh, in the foreword to A. Kaufmann's
Introduction to the theory of Fuzzy Subsets Vol. 1

In 1965, Loft A. Zadeh coined the term "fuzzy" as a formalization of the long-debated philosophical concept of "vagueness." This formalization has evolved into the theory of fuzzy subsets and fuzzy logic. The theory itself, though based on "fuzzy" or vague concepts, uses a precise mechanism that has all of the rigor found in the foundations of traditional mathematics. Similar to traditional mathematics, the foundations of "fuzzy" theory contains formalized logic, definition of sets and membership, as well as a description of operations.

1.1 Fuzzy Logic

Traditional propositional logic deals with propositions that are given truth values in $\{0,1\}$ and are symbolized by some variable, such as "p." Basic logical operations include: negation ($\sim$), conjunction ($\vee$), and disjunction ($\wedge$) each with its own truth table. The combination of these propositions with operations form more complicated "sentences" [5] .

Fuzzy logic is merely an extension of this basic logic. First, fuzzy logic allows for a proposition to have a truth value in the interval $[0,1]$ instead of restricting to $\{0,1\}$. So the truth value of p [$tv(p)$], for example, can be 0.1 where p is not entirely false because the truth of p is > 0 but is instead partially true, much like most linguistic sentences. This can mimic the statement "a 30 year-old is old." 30 is certainly not considered by most to be old, but a 15 year-old can consider this individual as "old." So it is not entirely false, but has a low truth value.

Negation. Negation is redefined to be $1-tv(p)$. If we apply this rule to propositional logic were $tv(p)$ restricted to 0 and 1, we find that $tv(\sim p) = 1-0 = 1$ and $tv(\sim p) = 1-1 = 0$ respectively. So, there is no tension with the previous operator.

Disjunction. Disjunction in fuzzy logic is given by the use of t-norms. There are many ways to form a t–norm but the most popular, historically, is given by:

$$T_m(a,b) = min(a,b) \tag{1}$$

Conjunction. Similarly, conjunction is given by a t-conorm; the most popular of which is given by:

$$C_m(a,b) = max(a,b) \tag{2}$$

where $a,b \in [0,1]$. And again we find that application of these rules to the traditional forms of propositions yield the same results as before. Using these operations, we can also create additional operations and sentences as before that will yield consistent results with previous definitions given the restrictions to 0 and 1 [5] .

1.2 Fuzzy Subsets[4]

A crisp, or a classical subset, is a "collection of distinct well-defined objects" [5] . A subset is generally denoted by a capital letter (A, B, $C\ldots$). An element is denoted by a lower case letter

[4] For further discussion about the mathematics of fuzzy logic and fuzzy set theory see A.Kaufmann's Introduction for the Theory of Fuzzy Subsets.

$(a, b, c \ldots)$. The membership of an element b to a subset B can be construed in terms of logic: $tv(b$ is a member of $B) = 1$ or 0; b is either an element of B or not. Here, fuzzy subsets depart from their crisp counterparts.

In a fuzzy subset $(A, U \ldots)$ membership is not simply 1 or 0. Just as truth values can take values in the interval $[0,1]$, membership can also be between 1 and 0. From the previous example, we can see that a 30 year-old is not a full member of the "set of old people," but it is also not entirely false, and therefore, not entirely not a member of the "set of old people." So, fuzzy subsets also have a defined "membership function" (μ) that determines to what degree an element is a member of a particular subset. A formal definition looks like the following [14]:

Let O be a set and let x be an element of O. Then a fuzzy subset U of O is a set of ordered pairs

$$\{(x \mid \mu_U(x))\}, \tag{3}$$

for all $x \in O$ where μ_U is a membership function taking its values from a membership set M.

If $M = \{0,1\}$, this set will simply be a classical set.

Definition 1.1 (Inclusion). In classical set theory, we say that a set A is included in a set U if for all $x \in A$, $x \in U$. Similarly, given a set E, an associated membership set $M = [0,1]$, and fuzzy subsets A and U, we can say that A is included in U if for all $x \in E$; $\mu_A(x) \leq \mu_U(x)$.

We can also define basic operations such as complementation, intersection and union; however, here their meanings are more significantly changed. For the following definitions, let E be a set and $M = [0,1]$ the associated membership set, and let A and U be fuzzy subsets of E.

Definition 1.2 (Complementation). A and U are complementary if for all $x \in E$;

$$1 - \mu A(x) = \mu U(x). \tag{4}$$

Note 1.1. without the $M = \{0,1\}$ restriction, the intersection of A and its compliment (A^c) may not be empty and the union will not always be E.

Definition 1.3 (Intersection). $(A \cap U)$ The intersection of A and U is defined to be "the largest fuzzy subset contained at the same time in A and U" [14] or more formally for all $x \in E$;

$$\mu_{A \cap U}(x) = \min(\mu_A(x), \mu_U(x)). \tag{5}$$

Definition 1.4 (Union). $(A \cup U)$ The union of A and U is defined to be the "smallest fuzzy subset that contains both A and U" [14]. In other words, for all $x \in E$;

$$\mu_{A \cup U}(x) = \max(\mu_A(x), \mu_U(x)). \tag{6}$$

Definition 1.5 (Properties). Using these definitions, we can explore the properties of these subsets. Though the proofs will not be given here, fuzzy subsets satisfy the same properties as that of a crisp set except for two: $A \cup A^c = E$ and $A \cap A^c = \varnothing$. The properties for fuzzy subsets are:

Commutativity
Associativity
Idempotence
Distributivity

$A \cap \varnothing = \varnothing$, where $\varnothing$ is defined in the usual way.
$A\ U\ \varnothing = A$
$A\ \cap E = A$
$A\ U\ E = E$
Involution
De Morgan's theorems

Thus, we can see that fuzzy subset theory and logic are simply extensions of classical theory. Though the basic unit is "fuzzy," the techniques of evaluation actually remain quite rigorous [14] , following the same mathematical procedure as classical models. However, since fuzzy subsets and fuzzy logic is an extension of classical ones, the types of problems these units can deal with are also expanded. There is now a way to deal with "fuzzy phrases" and use them in computation and modeling. This attribute of fuzzy subsets and fuzzy logic leads us to its application and usefulness in the field of economics.

1.3 Approximate Reasoning[5]

Approximate reasoning is defined as "the method of processing information through fuzzy rules" [5] . For example, if we have the fuzzy rule:

$$\textit{if } x \textit{ is } A, \textit{ then } y \textit{ is } I \tag{7}$$

where the A is a subset of a universal set **X** and I of universal set **Y**. Fuzzy sets are usually derived from linguistic variable and approximate reasoning would be used to evaluate the value of y given x.

For example, suppose we defined the first fuzzy set (A) to be size and the second (I) to be proximity. The phrase then would read something like

$$\textit{if } x \textit{ is small}, \textit{ then } y \textit{ is far away}. \tag{8}$$

Now, in order to make some sort of judgment about y given x, an implication operator P from classical logic is usually chosen giving us

$$(x \textit{ is } A) \rightarrow (y \textit{ is } I), \tag{9}$$

as our rule. Using this implication operator P, we can produce a fuzzy relation R on $\mathbf{X} \times \mathbf{Y}$ where R is defined by

$$R(x,y) = P[A(x), I(y)]. \tag{10}$$

We can also call this processes "fuzzy inference."

This reasoning can also be applied to blocks of multiple rules with a similar effect. Suppose A_i and I_i are fuzzy subsets of A and I respectively, where $1 \le i \le N$. In this case, in stead of just one equation

$$(x \textit{ is } A) \rightarrow (y \textit{ is } I), \tag{11}$$

we have a block of rules

[5] An Introduction to Fuzzy Logic and Fuzzy Sets

1. if x is A_1, then y is I_1
2. if x is A_2, then y is I_2

   ⋮

   for $1 \leq i \leq N$.

We then define a relation for each rule making a block or relations

$$i)\ R_i = P[A_i(x), I_i(y)], \tag{12}$$

where $1 \leq i \leq N$.

Now, to evaluate the block of fuzzy rules, we can either choose to

1. infer first and then aggregate the conclusions by finding the union of all concluded sets I_i (this union can be the t–conorm, C_m) or
2. aggregate first and then infer.

Hence, we can establish a version of this process into a "fuzzy inference algorithm" shown below [24] :

Find the degree, μ_i, to which rule i applies ie. $\mu_i = A_i(x^*)$ (the membership function of A_i)

Find the effective output of rule i as a fuzzy subset E_i where the degree of membership is defined by

$$E_i(y) = N(\mu_i, I_i(y)) \tag{13}$$

where $N(\mu_i, I_i(y))$ is usually implemented by a t–norm (T_m).

Find the overall model of output as fuzzy set E = union of Ei where the membership grade of E is determined as

$$E(y) = M(E_1(y), E_2(y), \ldots, E_n(y)) \tag{14}$$

4) Defuzzify E to obtain a crisp output value, y*.

This approximate reasoning method will be crucial in understanding how most of the following applications dealt with linguistic variables and arrived at their conclusions.

2 Economic Applications: Literature Survey

Fuzzy subset theory and fuzzy logic have been employed in various economics areas. In particular, it has been used to mimic the human thought process. Humans, being limited beings, do not have the time nor the necessary capabilities to process massive amounts of information in order to make a single decision. Instead, people use approximations, educated guesses, and estimations in order to aggregate information and attempt to make a rational decision.

So, instead of using a precise model that takes all available information into account when coming up with a conclusion, a fuzzy system can take only those bits of information that an average person would and construct a definitive model. These models have thus far been shown to be able to expand current models and simultaneously address several questions and situations that classical models are unable to conclusively answer and maintain consistency with previous models. Just a few potential applications are in the following areas explored in this paper: finance, international and national policy, microeconomic theory, and game theory. Each of these areas uses fuzzy theory and fuzzy systems to expand upon old ideas or even solve previously unsolvable problems.

In general, the creation of the fuzzy model follows these steps: fuzzification, inference, and defuzzification. First, data is transformed into subjective categories or vague linguistic variables and "fuzzified." Next, a set of fuzzy rules from membership functions of the inputs and determined outputs are formed based upon causality principles. Finally, at the defuzzification stage, linguistic terms are transformed back into real numbers for further processing. This process is essentially the approximate reasoning described in the previous section.

2.1 Economic Theory

Economic theory thus far has been fairly divided between linguistic and mathematical economists [16] . Though the integration of mathematical models in the past 20 years have been greatly increased, there is still a divide between linguistic descriptions and rationales with mathematical modeling. While modeling allows for precise solutions, they require quantifiable inputs that may or may not always be available and tend to miss the subtleties of human behavior that can be caught by linguistic descriptions.

However, linguistic descriptions have the potential to be mathematical thanks to the intensive research of the last half century in artificial intelligence. These developments use the long recognized philosophy of the vagueness principle. Consistent with these findings, Pfeilsticker contends that the basis of the economic method divide is the inherent vagueness of attributes and concepts describing human behavior and reasoning and proposes that fuzzy logic can be the "bridge connecting the two sides" [16] [21] . Here he describes in a few pages, the "sensibility" of appling fuzzy logic to describe economic concepts.

For example, how does one decide when a recession has happened? In crisp mathematics, it may be necessary to set some sort of numerical cutoff where something abruptly becomes a "recession." However, it is clear that this method is unreasonable and borders on ridiculous. Instead, there should be some sort of spectrum deciding when something is a recession. Pfeilsticker proposes something like

$$\mu_{REC} = \frac{2\mu_{GNP}(x) + \mu_{inv}(x) + \mu_{une}(x) + \mu_{nin}(x)}{5} \tag{15}$$

where μ_{REC} is the membership function related to the recession and $\mu_{inv}(x)$, $\mu_{une}(x)$, $\mu_{nin}(x)$ are increasing membership functions related to months of declining investments, months of declining national income, and months of increasing unemployment respectively. This will give us a range of values to suggest how "bad" a recession is. This way, it is also possible to represent a depression as a "heavy recession" by manipulating the formula above. For example, he defines it as

$$\mu_{DEP}(x) = (\mu_{REC}(x-3))^2 \tag{16}$$

where μ_{DEP} is the degree of the depression [16].

In light of the potential usefulness of fuzzy logic in theoretical economics, below are a few other applications to theoretical models.

2.2 Utility Theory

A key assumption in economics is the ability of a set of economic choices to be ordered by an economic agent according to some ranking of preferences. However, since preferences are based on human reasoning and the causality principle associated with the way human minds process information in chunks, they are inherently vague. Simply state, the economic agent may not actually know exactly what his or her preferences are, so a set ranking is almost impossible. However, one can develop a sort of relation that can describe such a preference structure. The one proposed by Ponsard characterizes the behavior by a structure which he denotes as $(\mathbf{X}, R)$, where R is a fuzzy relation between the elements of $\mathbf{X}^2$. Another words

$$R = \{(x_i, x_j),\ \mu_R;\ \text{for all } x_i \in \mathbf{X} : \mu_R(x_i, x_j) \in M\} \tag{17}$$

where the membership set M is a lattice [18] .

Using this relation, a preference structure can be described between any number of relations. For example, Ponsard defines

$$\mu_R(x_i, x_j) > \mu_R(x_j, x_i) \tag{18}$$

to mean that there is a strong degree of preference for x_i relative to x_j. This particular definition has the property of Max-Min transitivity where the "direct preference between two possible alternatives is at least as strong as the indirect preferences which require the intervention of a third possible alternative" [18] .Also, when an agent is indifferent, the have the property of symmetry:

$$\forall (x_i, x_j) \in \mathbf{X}^2,\ \mu_R(x_i, x_j) = \mu_R(x_j, x_{i.}). \tag{19}$$

Using these definitions and demonstrating that the "existence of a utility function implies the existence of a continuous function inside a fuzzy topology" [3] . Ponsard has proven that given certain conditions, we can guarantee the existence and continuity of a fuzzy utility function in the case of a countable set of objects.

This conclusion was then extended by Billot [5] . He demonstrates the order-preserving property of the definable continuous utility function. Additionally, he concludes that the mathematical conclusions reached from such an analysis are actually an extension of classical economic theory. In fact, he states that "there is an implicit fuzzy model" within traditional models that essentially make the traditional models a "special case" of the fuzzy one [3] .

2.3 Equilibrium Theory

Now that the case for the existence of a utility function has been established, one can move onto the existence of equilibria. However, the beauty of the traditional model, really could be because of the ease in finding the equilibrium. Since, fuzzy logic tends to initially complicate things a bit, finding the equilibrium is a bit more involved. But, by definition it still starts with supply = demand [24] .

According to Yager, if we take a look at the equilibrium where y is piece-wise linear, then we can find a definite equilibrium. Given that the supply is modeled by the rules

$$\textit{if } U \textit{ is } A_i,\ \textit{then } V \textit{ is } g_i(u),\ i = 1, 2, \ldots, n \tag{20}$$

where g is supply vs. price and

$$\textit{if } U \textit{ is } B_j,\ \textit{then } W \textit{ is } h_j(u),\ j = 1, 2, \ldots, r \tag{21}$$

where h is demand vs. price.

This means that our solution to our supply and demand functions look lik

$$y = \frac{\sum_{i=1}^{n} A_i(x) g_i(x)}{\sum_{i=1}^{r} A_i(x)} \tag{22}$$

and

$$z = \frac{\sum_{j=1}^{r} B_j(x) h_j(x)}{\sum_{j=1}^{r} B_j(x)} \tag{23}$$

respectively. So, the equilibrium must be where $y = z$. Here, Yager claims that given the piecewise linearity of A_i and B_j the equation has a solution Rk that is a range of prices that could be an equilibrium. Rk is derived from the union of all possible solutions from supply and demand given a certain rule I [24] .

Yager's finding is actually based off of a previous paper by Ponsard, who found a general economic equilibrium assuming the continuity of the demand vs. price function [18] . He begins by defining the initial wealth in the economy as

$$w = \sum_{j=1}^{m} w_j \tag{24}$$

where w_j is the "vector of the j–th consumer resources" [18] . Supposing some initial distribution of stocks and the equal distribution of profit between consumers = 1 in the aggregate for the r firms involved, we can get an idea of where the consumer's income is coming from. Now, let

$$x = \sum_{j=1}^{m} x_j \tag{25}$$

be the total consumption and

$$X = \sum_{j=1}^{m} X_j \tag{26}$$

be the total consumption set.

Now, he subjectively uses the median to aggregate the membership function outputs so we get

$$\mu_x(x) = Median[\sup \mu_{xi}(x_j)], \tag{27}$$

for $j = 1, \ldots, m$ and $x_j \in X_j$ where μ_x is the membership function of the individual consumptions from X to $[0,1]$.

So the aggregation describes the behavior of the median consumer.

Similarly, we get

$$\mu_y(y) = Median[\sup \mu_{yk}(y_k)], \tag{28}$$

for $k = 1, \ldots, r$ and $y_k \in Y_k$ where we now get the median producer's behavior.

From here we can get the excess demand

$$e = x - y - w \tag{29}$$

for $x \in X$ and $y \in Y$. The element e can also be denoted as part of a set E defined

$$E = X - Y - \{w\}. \tag{30}$$

e is also an element of the fuzzy subset E (which is a subset of E). And a membership function μ_E is defined from E to $[0,1]$ and dependent of $(x-y)$ such that $\mu_E(e)=0$ when $(x-y)\leq 0$ and $\mu_E(e)=1$ when $y=0$.

Given this construction and an excess demand point-to-set mapping from P (the set of prices) to E previously called a function g (here it is assumed to be continuous), we can use the famous Walrus Law, which states that "excess demands must be non-positive and the goods whose excess demand is negative must have null price [18] . Using this law and an adjustment to all prices we can find a fixed point that corresponds to the competitive equilibrium state [18] .

2.4 Oligopoly Markets

Oligopoly markets, unlike pure competition markets and monopolies, are said to inefficient. Part of this inefficiency is due to behavioral uncertainty stemming from interdependencies between multiple firms where a behavior change in one firm can affect the entire market, a feature absent in the two markets mentioned previously [12] . In addition to this behavioral uncertainty, oligopoly markets, by definition are vague concepts. They are supposed to be a market controlled by a "few" firms that make similar products. However, what does few actually entail? If a discrete number like 4 is given, does that mean that 5 is no longer few? If not, then how does one pick a value to be few?

Greenhut, Mansur and Temponi address these questions by applying fuzzy sets to the oligopoly model. In addition to "few," these authors also point out the inherent vagueness of concepts like "similar product" and "interdependent." So, in order to deal with these concepts, the authors have established fuzzy membership functions to determine the degree of membership of a value or product. Using fuzzy arithmetic, they developed an algorithm that can "guesstimate" the cost of oligopolic behavioral uncertainty.

2.5 Game Theory/Negotiation

One of the most widely used methods in attempting to model human behavior in the social sciences is game theory. The basic idea behind game theory is that real world interactions can be formulates as a "game." Each of these games consists of a description of the players, player actions and preferences, payoffs and game structure. The players in these games make strategic decisions based on their knowledge and beliefs of the components of the game in order to increase their payoff. Just from looking at this basic framework, it is easy to see several areas where information may not be precise and can be modeled as a fuzzy variable.

In a widely referenced article about fuzzy games, Butnariu [5] notes that the classic 2-person model of a game conceived by Morgenstern and von Neumann[6] assumes "equally possible choices" for each player in the game. However, this is frequently demonstrated to not be the case. People, in general, do not make optimal choices but "best alternatives." People consider all implications of an action including risk, beliefs about and of the other player, and so on. So the alternatives can be considered as a member of a fuzzy set of "feasible options" with a certain degree of membership, essentially allowing for fuzzy preferences [5] .

For an n-person game when cooperation is possible, a coalition can form. However, unlike the traditional form of an n-person game where a coalition if formed around the unquestioned acceptance of a particular stance, a fuzzy coalition can be used to demonstrate a partial acceptance of a stance.

[6] In this model, players can have both pure and mixed strategies delineated by a probability distribution over a set of potential actions. The expected utility of such a strategy is considered to be "payoff."

Butnariu proposes a fuzzy set to describe the grade of acceptance of a stance as well as a fuzzy relation between the different members. This he calls an n-person fuzzy game [5] .

To extend upon this idea, Garagic and Cruz [12] have proposed a way to incorporate fuzzy set theory to study n-person non-cooperative games. In general game theory, an n-person non-cooperative game would require a set of actions, a payoff function and something to represent a preference ordering, and in order to incorporate some beliefs about other players, one would have to tweak the payoff functions or preferences to represent that. Much of the "tweaking" in theses situations depend on the desired result. However, Garagic and Cruz propose a way to model a player's knowledge about decision-making strategies into the game using fuzzy sets without specifically changing payoff functions. Garagic and Cruz's processes include fuzzification, inference and defuzzification.

The fuzzification step is used to describe the uncertainty behind a player's decision to play a particular strategy. Garagic and Cruz use a set of linguistic terms to describe the capacity of a given strategy such as {small, medium, large}. The strategies are then associated with a number in $[0, 1]$ by a membership function that describes to what extent that strategy has small/medium/large capacity.

Then, a set of rules is placed that provides a set of payoffs for each set of strategies and a "fuzzy logic quantification" of the linguistic descriptions used forming a rule base. Fuzzy inference is built on top of the rule base to "allow each player to incorporate his/her heuristic knowledge of a possible intent of the other player" into his/her strategy by a mapping to his/her preferences [12] . And, the defuzzification step produces a crisp number from the output. In this case, the number is a cardinal measure of a player's preference suggesting which strategy is most feasible considering the other player's preferences.

Using this process, the authors have been able to conclude the existence of at least one Nash Equilibrium for an n-person non-cooperative fuzzy game. They have demonstrated that using fuzzy constraints instead of crisp ones cut down on the number constraints and reduce the complexity of structuring the game. Thus, they have been able to produce a smaller set of potential solutions and somewhat simplified a "complex multidecision-making problem" [12] .

The literature in this area is quite large, for example, Song and Kandel [20] studies a variation of the prisoner's dilemma where a player's desire to hurt or help his/her partner is fuzzy. There is great potential for extensions in game theory, and these are just a few things that have been explored. However, in addition to direct applications to game theory, there are also applications to areas traditionally modeled by game theory.

For example, Wasfy and Hosni's [23] paper creates the framework for a negotiation simulation. Game theoretic models have been used to try to determine the equilibrium outcome for a negotiation; however, the factors in a negotiation tend to be imprecise and large in number. Thus the model they have developed harness the power of fuzzy subsets and logic to describe the interaction between external factors (time constraints, other potential competitors. . .) and internal power factors (concession force, resistance force, commitment. . .) that determines how the negation plays out. They created strategic profiles reflecting resistance power by observing and talking to negotiators and categorized the types of strategies that can be played. They also used fuzzy logic in order to calculate their two biggest determinants: concession force and resistance force. The variables are all attached to fuzzy membership functions and set to influence either resistance or concession force and is accounted for in that manner. The potential for use in negotiation simulations are great and the possibilities for tweaking the program to incorporate more variables are also promising.

2.6 Practical Applications (Finance and Policy)

Arguably, the most important use of any model is its applicability for decision-making. In economics, the two most prominent decision-making tasks are in finance and economic policy. Financial applications include: valuation, pricing, and credit risk analysis among other things. Policy applications can be both international in nature (international trade, international coordination found in summitry. . .) and country or region-specific (sustainability assessment, underground economies, cost-effectiveness of projects, monetary policy. . .).

2.7 Finance

In general, one does not think of finance being fuzzy. However, there are occasions when a lack of information makes fuzzy theory incredibly relevant. In terms of traditional financial theory, a key article was written by Buckley concerning the “fuzzy mathematics of finance” [4] .

Basic finance deals with variables such as time, cash, and the interest rate. In traditional financial calculations, these numbers are generally set at some level that is estimated based on previous knowledge. However, what this says is that these numbers are inherently fuzzy. Thus, they can be modeled as fuzzy sets; when numbers are made into fuzzy subsets of real numbers, they are called fuzzy numbers. This process is called “fuzzification.”

Now, based on these values, traditional finance has calculated future and present value in order to make financial decisions. These can also be fuzzified and used in financial decision-making by ranking fuzzy investment alternatives [4] .

Looking at Buckley’s work, we see that if we consider $S_n = A(1+r)^n$ to be the amount of cash in an account after n periods (where A is the amount invested today at rate r per period for n months), we can find a fuzzy S_n defined

$$\underline{S_n} = \underline{A} * (1+r)^n \tag{31}$$

where the operations $*$ and $+$, in this case, are fuzzy operators because we are doing arithmetic with fuzzy numbers [4] .

Now, if we also make the number of time periods n fuzzy ($\underline{n}$), then we get a fuzzy function $\underline{S}$ where the membership function is generally defined by

$$\mu(x|\underline{S}) = \sup \Gamma(x)(\Theta) \tag{32}$$

where $\mu(u|\underline{A})$ denotes the membership of a fuzzy set $\underline{A}$, $\Theta = \min[\mu(u|\underline{A}), \mu(v|\underline{r}), \mu(w|\underline{n})]$ and $\Gamma(x) = \{(u,v,x)|u(1+v)^w = x\}$. This means that if fuzzy number $n = \underline{n}$ then the membership function would reduce from $\mu(x|\underline{S})$ to $\mu(x|S_n)$ [4] .

We can also do the same for a compounded interest rate. For example, given that we were given a rate r with m compoundings, we would simply get $r*\frac{1}{m}$ as the periodic rate and mn as the number of periods. Given that it is continuously compounded, we sent m to $+\infty$ to get $A*(1+e)^n$ where the calculation for e is very similar to the crisp equation $e = (1+\frac{r}{m})^m - 1$.

Now turning to present value, we know that the crisp version of the present value of a future cash amount S is

$$PV(S) = S(1+r)^{-n}. \tag{33}$$

According to Buckley, we need two definitions of $PV(S,n)$ for fuzzy amount S, and periods n at the fuzzy rate r.

1. $PV1(S,n) = A$ iff A is a fuzzy number and $A * (1+r)^n = S$
2. $PV2(S,n) = A$ iff A is a fuzzy number and $A = S * (1+r)^{-n}$

where the first is for when S is negative and the second for when it is positive. This way the present value is always defined.

From here we can develop definitions for annuities and cash flows that can be used to calculate the net present value (NPV) of a set of projects. In crisp financial analysis, NPV can be used to rank the attractiveness of a set of projects. Similarly, though using a slightly modified method of comparison, Buckley has also discussed a way to rank fuzzy NPVs.

Given above is simply the bare bones of basic financial analysis from which everything else is derived; however, further work has also been done by people like Castillo and Melina about time series of consumer goods [8] , Tay and Linn on fuzzy reasoning and security prices [22] , and Alaves on Credit Risk analysis [1] . Each has been able to demonstrate the applicability of fuzzy logic in financial analysis.

2.8 Policy

International The biggest concerns in international economic policies center around trade. The greatest problem in modeling trade is based on the need in traditional economic models to use continuous, if not linear, functions in order to model country behavior. This calls for highly idealized situations that fail to satisfactorily model economic trade interactions. These linear models cannot handle the dynamic quality sometimes found in international trade relations. This problem could be solved if there was a way to handle such dynamics. One proposed method, presented by Castillo and Melin, suggests a non-linear mathematical model that utilizes genetic algorithms to describe a "best fit" for variable parameters and a fuzzy-rule base for "behavior-identification" [9] . They have proposed that this system can be an intelligent system used to forecast the future behavior of international trade for a set of given countries. Considering that the model is actually trying to mimic the reasoning process of human experts in the financial field, it is a promising tool for industrial or government decision-making processes.

Traditional modeling utilizes linear statistical models from times series Econometric Theory. Utilizing the Keynesian Model of a single economy described by

$$\begin{aligned} I = I(Y,r),\ Iy > 0,\ Ir < 0 \\ S = S(Y,r),\ Sy > 0,\ Sr < 0\text{'} \end{aligned} \tag{34}$$

where Y is income, r is the interest rate, M is a constant monetary supply, I is investment, S is savings and assuming a fixed price P. Also, let a function $L(Y,r)$ denote the liquidity preference where $Ly > 0$ and $Lr < 0$[9] .

Genetic algorithms were used to find "best fit" parameters (α, β below) for the given functions behavior identification models could be defined with a fuzzy rule base [9] . For example, the authors give this example for a bi-dimensional autonomous model:

$$X = \alpha\ f(x,y) \tag{35}$$

$$Y = \beta\ g(x,y) \tag{36}$$

where the equilibrium(x^*,y^*) is considered to be stable if:

$$\alpha\ f_x + (g_y - \beta) < 0. \tag{37}$$

This can be stated in fuzzy logic as:

$$\text{IF } [\alpha f_x + (g_y - \beta) < 0] \text{ THEN } Equilibria = stable \tag{38}$$

In simulations for the case of USA, Mexico and Canada demonstrated the dynamic behavior of investment in the international trade system as Mexico moved from a closed to open economy, dynamics that tend to be lost in traditional econometric analyses. The model generated a potential future trade dynamic after the transition from closed to open economy was made demonstrating its possible application in industrial and government decision-making [9] .

On a similar note, when considering international trade, businesses have to consider international conditions when considering the effects of expanding internationally. Levy and Yoon have proposed a method to determine whether one should enter or not given political, social and economic characteristics of a region or country [15] . This paper also uses the similar fuzzification, inference, and defuzzification rout.

The initial fuzzification of variables such as GDP growth rate, exchange rates, sales potential as well as political risk and market opportunity were assigned membership functions. For ease of calculation, they used a trapezoid to represent relevant fuzzy sets of global market entry. This method has been determined to be the most efficient. These he also created into linguistic variables with hedges, giving them values such as {highly negative, slightly negative, slightly positive, highly positive} or others as they saw appropriate.

Their analysis included a block of fuzzy rules that determined which set of rules were the most applicable and related. And the final step, as always, is the defuzzification step where a set decision was announced. These methods developed a "decision framework for fuzzy logic to model the major categories of" global market entry analysis [15] . The work done here could easily be used to compare with past decisions and changed in order to better fit human experience. However, it is still a potentially valuable model for further decisions on international market entry, a key concern in this day and age.

National/Regional Though international economics and policy have recently taken a bigger spot light in policy debates, the issue of national policy is still a major concern in today's society. There will always be national and region specific situations that can also be dealt with using fuzzy theory. A few examples would be: coral reef policies, underground economies, and unemployment [19] [10] [7] .

The coral reef is an immensely intricate system. The interactions between the reef, the surrounding areas, and living organisms is still not yet entirely understood, but the concern for policy makers is how to make cost-effective decisions in order to preserve the coral reef and maintain efficiency. The truth of the matter is that since we cannot understand how such a complex system operates; there really is no way to effectively and accurately model the system. Moreover, given that we do not know all of the variables involved, we thus cannot make an "optimal decision." All decisions will inherently be "sub-optimal" due to these constraints. However, the point is to optimize the sub-optimal by recognizing the more important variables and achieving a more reliable mathematical model to aid in decision-making [19].

Here the authors use the method of fuzzification, inference, and defuzzification in order to develop their model which included both the reef impact and economic policy in order to find the most cost-effective. They consider nutrient and sediment influx, physical oceanographic characteristics, and biotic state variable in their inputs [19] . They fuzzify these inputs by putting a linguistic range on them {low, medium, high} and a membership function with them, making them fuzzy sets. Then they set a list of fuzzy rules and following fuzzy inference rules, they can draw conclusions. They used the method of "scaling" to determine how much of what rules apply and how to determine which was the most applicable. The final step is defuzzification, at which time the fuzzy output is translated into a crisp number that can be used for policy concerns.

This particular model, the authors claim, provides a dynamic forecasting system to replace the static, linear one previously used. They can incorporate both the non-linear economic payoff functions and the model for the complex interactions of the coral reef in order to determine what a "well-maintained reef" is.

In the same light, sustainability was also an economic policy that had been selected for study with fuzzy logic [17] . The inherent fuzziness behind the concept of "sustainable" made this an easy target. Sticking to the preferred methodology, the article chooses several important fuzzy linguistic inputs some of which being: health, water integrity, air integrity and policy priorities. Using these inputs and a set of fuzzy IF-THEN rules, they were able to develop a model that combined economic and ecological priorities instead of just one or the other, as more traditional models would have done. Furthermore, the fuzzy nature of the program allows it to work without clear data for the inputs and the outputs are easy to interpret, making this program (SAFE) ideal for policy-makers.

On a different note, other authors have used fuzzy sets to set up a time series in order to determine the behavior of other data-lacking areas. For example, Draeseke and Giles have created a model for New Zealand's underground economy [10] . For their method, they chose to use 2 variables instead of the 10 used by their linear, traditional counterparts. Also, with limited information on the quantity and quality of the interactions between variables, minimizing such variables could help settle at least one problem. Then, this paper also set some fuzzy inference rules and drew a conclusion about the size and activity of the underground economy. Though there is no real way to test these conclusions, according to empirical graphical comparisons, while the traditional model demonstrates the trend rather well, it fails to mimic the magnitudes that were well captured by the fuzzy model.

Also, it is possible for authors to measure subjective measures, such as consumer confidence, by objective levels given by inflation of unemployment [7] . It is not a stretch to believe that consumer confidence is highly linked to the consumer's perceptions about the economy according to indicators. Traditionally, European economic policies spent a great deal of time trying to control inflation rates, believing them to be the greatest influence on the performance of the economy; however, this paper has found that unemployment is also very important.

Again the method of analysis comes down to fuzzificaiton, inference and defuzzification. They accurately assume that people cannot accurately absorb and process all of the information necessary to make an informed decision, so they have attempted to mimic a person's decision making capabilities by fuzzifying the indication of observable phenomena to {strong, weak} or {high, medium, low} as the case may be. According to their analyses, unemployment holds a much higher sway on consumer confidence than inflation. Thus, the authors believe that it would wise for the EU to consider such things in their policy meeting[7].

[7] A similar study for such international cooperation has also been done in realation to summits and its effects. They were also able to model and determine the effects these summit resolutions had on behavior and efficacy. Previous methods had come up inconclusive, but these came to the conclusion that while benefits are hard to measure, degrees of compliance is possible to measure [2].

In addition to adding new insights into the operations of a system, policy applications have also been useful in determining important variables and factors in analyses in areas where traditional methods have come up as inconclusive. In light of these progressions, introduction of new methodologies in decision making processes outside of the production management field is clearly crucial.

3 Closing Remarks

The ability of fuzzy methodology to allow linguistic variables to be mathematically modeled is a great advantage for any decision-making process,[8] and economic applications of fuzzy subset theory and fuzzy logic have been explored since the 1970s. Many of these applications were initially in production and operations management, but have since then expanded to cover many other areas of economics. The readings presented were only a glimpse of the wide literature of economics applications of mathematics of uncertainty, much less, of fuzzy modeling. However, even from these few readings, it is evident that fuzzy mathematical models hold great promise in expanding upon traditional economic models, allowing economics and policy makers to tackle previously unsolvable problems.

In an increasingly demanding world, decision-making potential have come to the forefront of academic and industrial research alike. The push for more decision-making capabilities will increase the demand for solutions, some of which cannot be adequately provided (if at all) by traditional mathematical models. Furthermore, fuzzy theory can handle much more complex, complicated, and interdependent economic problems through approximated reasoning. Thus, continued research into other modeling techniques will be necessary. Fuzzy subsets provide a rigorous mathematical foundation and valuable extension for existing models and, most likely, for those to follow. Thus, it is this student's belief that fuzzy theory, and those like it, will be, and even ought to be, integrated into mainstream academia. Probability theory and statistics took nearly 400 years to be considered a legitimate branch of study; in this fast-pasted, solution-oriented society, it will be necessary for a promising new branch of study to be integrated into the mainstream much faster. So, why not now?

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[8] This ability is also a very significant foundation of artificial intelligence, initially used in classical first and second order logic and now in the fuzzy logic framework (Wang).

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On Vagueness and Bipolarity

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Abstract. Classical logic is based on the principle of mathematical abstraction in crisp set theory which claims that the concept of element is self-evident without the need of proof. Furthermore, the properties of a set are usually considered completely independent of the nature of its elements. This rarely challenged and so-called "unchallengeable" doctrine in classical set theory, however, is arguably problematic. This paper reexamines the limitations of this very basic assumption. It is agued that Zadeh's fuzzy set theory presents a successful challenge to this very doctrine. Now the equilibrium-based bipolar crisp/fuzzy set theory has opened a double fronts in its challenge to the "unchallengeable." Philosophical aspects of the ongoing developments are discussed in this paper. A borderline of isomorphism is identified.

Index Terms— Vagueness; Bipolarity; Mathematical Abstraction; Isomorphism; A Lengendary Story

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1 Introduction

In mathematics, a set can be thought of as any collection of distinct objects considered as a whole. Although this appears to be a simple idea, set is one of the most fundamental concepts in modern mathematics. The principle of mathematical abstraction in classical set theory was rarely challenged throughout the history. The principle claims that the concept of element is self-evident without the need of proof of any kind and the properties of a set are independent of the nature of its elements. Classical logic is based on this principle. Thus, a classical (crisp) set X in a universe U can be defined in the form of its characterization function $\mu_X : U \to \{0, 1\}$ which yields the value "1" or "true" for the elements belong to the set X and "0" or "false" for the elements excluded from the set X. Evidently, the classical set theory is based on Aristotle's universe of truth objects as defined in the unipolar space $\{0, 1\}$.

Since the principle has been rarely questioned in the history it is commonly considered "unquestionable". It does, however, have obvious limitations that can be and should be reexamined. For instance, it is hard to say that the property of a community as a set of people is independent of the nature of each element in the set; it is also hard to see the property of a biological agent with a set of genes is independent of the properties of the individual genes.

This paper is set out to reexamine the limitations of the fundamental principle or doctrine. It is argued that fuzzy set theory has already successfully challenged the doctrine. It is further argued that equilibrium-based bipolar crisp/fuzzy set theory has opened another frontier in challenging the "unchallengeable." In Section 2 we refresh some salient features of fuzzy set theory and its philosophical origins. In Section 3 we introduce YinYang bipolar set theory with some discussion on crisp bipolarity and it generalized fuzzy bipolarity. A major shift and refreshing view of vagueness and bipolarity is revisited.

2 2. The Philosophical Origins of Fuzzy Sets

The fuzzy set and logic research community successfully fought the decisive battles in seeking the recognition in the last half century with many dramatic stories. A fuzzy set (Zadeh, 1965) F in a universe U is defined in the form of its characterization function μ_F:$U \rightarrow [0,1]$. Since the fuzziness property of a fuzzy set is dependent on the fuzziness property of its elements, fuzzy set presents a most serious and successful challenge to the doctrine of mathematical abstraction. Among the different properties of a fuzzy set that depends on its elements there have been type 1, type 2, numerical, and linguistic fuzziness, respectively. That is, if some or all elements are fuzzy in nature a set is a fuzzy set in general.

While Professor Zadeh, the founder of fuzzy set theory, is undoubtedly the most qualified author to address the philosophical issues of fuzzy sets, we make a few observations regarding the philosophical origins of fuzzy sets.

Observation 1. It is observed that fuzzification in fuzzy sets preserves the truth-based property of classical set theory with the addition of infinite number of truth gray levels which can be defuzzified to 0 or 1. Therefore, fuzzy sets can be deemed an extension of classical set theory that is based largely on Aristotle's philosophy of science where the universe consists of a set of truth objects. The fuzzy extension is realized by fuzzifying the bivalent truth space or lattice from $\{0,1\}$ to $[0,1]$.

Observation 2. Fuzziness is obviously a synonym of vagueness and therefore fuzzy sets can be considered the mathematical foundation of vague philosophy which was founded by the 19^{th} century American empirical psychologist William James (Gavin, 1992; Keefe and Smith (Ed.) 1996; Wang, (Ed.), 2001).

Observation 3. Anticipating William James by a century the English philosopher David Hume is considered the first modern empirical psychologist (Shouse, 1952). Among David Hume's criticisms on Aristotle's philosophy of science the most famous (historically) is the critique of the principle of causality (David Hume, 1711– 1776) (Wikipedia). The founder of fuzzy sets Lotfi Zadeh has continued the criticism and become the strongest critic on causality principle nowadays (Zadeh, 2001). On the other hand, computation with words in fuzzy set representation can be considered an empirical approach to artificial intelligence because words in natural language are largely empirical.

Observation 4. In addition to observations 1-3, it should be pointed out that vagueness as an important topic was also discussed by many influential philosophers from Aristotle to Bertrand Russel. Therefore, the philosophical origin of fuzzy set theory include (among others) a fusion of Aristotle's philosophy of science, William James vague philosophy, and David Hume's imperialism. In addition, it can be observed that both fuzzy sets and intuitionism do not support the law of excluded middle. It is suggested that Vague sets are intuitionistic fuzzy sets (Bustince and Burillo 1996).

3 On The Philosophical Origin of YinYang Bipolarity

YinYang bipolar set is defined as a collection of bipolar equilibria (including full- quasi- or non-equilibria) each of which has a negative pole and positive pole. Thus, YinYang bipolar set is to "equilibrium-based" as classical set is to "truth-based." One is concerned with equilibrium and non-equilibrium; the other is concerned with truth and falsity.

Since any dynamic agent or system can be considered an equilibrium or non-equilibrium with bipolar equilibrium/non-equilibrium as a generic form, a bipolar set hence is dynamic in nature.

While the philosophical origin of fuzzy set theory may need further study, the philosophical origin of equilibrium-based bipolar sets is undoubted the YinYang theory. The reasons behind this assertion include: (1) bipolar sets are sets of bipolar equilibria; (2) YinYang is inarguably the oldest philosophy of equilibrium and harmony. Its root can be traced back more than 6,000 years before the first Chinese empire existed. It was recorded along with the I Ching (Yi Jing) (The Book of Changes) as a foundation of the Doist philosophy and religion. It is shown in (Zhang and Wang, 2007) that bipolar YinYang also bridges two gaps: (a) the philosophical gap between Aristotle and his teacher Plato and (b) the computational gap between logic and mathematics.

With bipolarity the two poles of a bipolar equilibrium can be considered as the fusion or binding of two truth objects with opposite polarity. Furthermore, equilibrium and universe form a philosophical "chicken and egg" paradox because no one really knows which one created the other in the very beginning. Despite the paradoxical issue, equilibrium or non-equilibrium is a well-known scientific physical concept because it is central concept in thermodynamics – the ultimate physical source of energy, life, and then existence. Furthermore, equilibrium seems to be the sole physical concept that can match the mighty universe to form a philosophical "chicken and egg" paradox.

An YinYang bipolar crisp set X (Zhang, 2005a) in a universe U of bipolar equilibria has been defined in the form of its characterization function $\mu_X : U \Rightarrow B_1$, where $B_1 = \{-1,0\} \times \{0,1\} = \{(-1,0),(0,0),(0,1),(-1,1)\}$, $(-1,1)$ stands for (negative pole true, positive pole true) or a balanced state of an equilibrium; $(-1,0)$ stands for (negative pole true, positive pole false) or an imbalanced state of an equilibrium; $(0,1)$ stands for (negative pole false, positive pole true) or another imbalanced state of an equilibrium; and $(0,0)$ stands for bipolar false or eternal equilibrium. Evidently, YinYang bipolar crisp set theory assumes a symmetrical bipolar universe of dynamic equilibria where everything including the universe itself is an equilibrium with bipolar equilibrium as its generic form.

A YinYang bipolar fuzzy set X (Zhang, 2006a) in a universe U of bipolar fuzzy equilibria has been be defined in the form of its characterization function $\mu_X : U \Rightarrow B_F$, where $B_F = [-1,0] \times [0,1]$. Evidently, bipolar fuzzy set is to bipolar crisp set as Zadeh's fuzzy set is to classical crisp set. One is a bipolar generalization and another is a unipolar extension. From a different perspective, bipolar fuzzy sets are to bipolar crisp sets as quasi-equilibria are to equilibria.

While fuzzy set theory has successfully challenged the doctrine in classical set theory, equilibrium-based YinYang bipolar sets (crisp or fuzzy) have so far "fought" a difficult uphill battle. The difficulty is due to the double frontier:

(1) YinYang bipolar crisp set theory has to overcome the limitation of the principle of mathematical abstraction in classical set theory with overwhelming rejection.

(2) YinYang bipolar fuzzy set theory has to overcome the truth-based unipolar heritage of classical fuzzy sets with paramount resistance.

4 Crisp Bipolarity

The concept of equilibrium is central in both social and natural sciences, especially in biological science, as manifested by the fundamental laws of thermodynamics and Nash equilibrium in macroeconomics (Nash, 1950,1951). While Boolean logic (Boole, 1854) and its crisp extensions provide a basis for logical computation and modern computer technology, they can not be directly used for bipolar knowledge representation in an open-world of non-linear dynamic equilibria/non-equilibria in mental health, multiagent systems, macroeconomics, global regulation and decision, biological/neurological systems, genomics, biomedicine, nanoscience, thermodynamics, sociology, market analysis, business management, artificial intelligence, and other domains (Zhang, et al, 1989-2007, Shi, et al, 1991; Ai, et al, 2000; Kim, et al, 2007)). This problem can be attributed to the fundamental limitation of classical bivalent logic. To overcome this limitation, belief revision (Alchourròn, Gärdenfors, and Makinson, 1985), truth maintenance (Doyle, 1979) and dynamic logic systems ([?,?];[?,?];Harel, Kozen, and Tiuryn, 2000 have been proposed. These approaches, however, are all based on the same unipolar bivalent lattice $\{0,1\}$ that does not support bipolar interaction.

The above limitations can be traced back to its origin - the principle of mathematical abstraction in classical set theory which claims that the concept of element is self-evident without the need for proof and the properties of a set are independent of the nature of its elements. It can be argued, however, this doctrine has a number of limitations for dynamic logical reasoning. It is evident that if we consider a collection of biological, genetic, or any dynamic agents or objects as a dynamic set, the property of the set has to be dependent of the nature of its elements such as some human diseases are traced to genetic heritage. Then, we have a dilemma. This dilemma can be called the unipolar bottleneck as characterized in the following:

- Firstly, all dynamic agents/systems can be in an equilibrium or non-equilibrium state and bipolar equilibrium is the generic form of multidimensional equilibrium; equilibrium-based reasoning should be holistic in nature where global equilibrium depends on local equilibrium/non-equilibrium. Therefore, the principle of mathematical abstraction in classical set theory does not apply.
- Secondly, the truth-based bivalent lattice $\{0,1\}$ is inadequate for characterizing a bipolar equilibrium due to its lack of bipolarity. With a truth-based system we face the dilemma: (a) if we treat each pole of a bipolar equilibrium as a self-evident element we do not have bipolar coexistence for equilibrium-based holistic reasoning; (b) if we treat a bipolar equilibrium as a self-evident element it can only be true or false. It can be observed, however, that a bipolar equilibrium could be in any of the four different crisp states: (i) $(0,0)$ – eternal equilibrium or both sides false; (ii) $(-1,0)$ – unbalance or negative side true and positive side false; (iii) $(0,1)$ – another unbalance or negative false and positive side true; (iv) $(-1,1)$ – balanced or both sides true.

The four different states of an equilibrium/non-equilibrium (such as mental equilibrium/non-equilibrium) constitute a bipolar lattice $\{-1,0\} \times \{0,1\}$ (Zhang, 2005, 2007; Zhang, Pandurangi, and Peace, 2007). It can be observed that bivalent logic can not be directly used for reasoning with bipolar equilibrium problems (Zhang, Pandurangi, and Peace, 2007). This can be illustrated with the following examples:

- A depressed patient took a positive antidepressant drug and regained mental equilibrium.
- Another patient took the same drug but became manic.

BUMP: $(\varphi^-,\varphi^+)_*(\psi^-,\psi^+)$,
$[((\varphi^-,\varphi^+) \Rightarrow (\phi^-,\phi^+)) \equiv (-\mathbf{1},\mathbf{1})\ \&\ ((\psi^-,\psi^+) \Rightarrow (\chi^-,\chi^+)) \equiv (-\mathbf{1},\mathbf{1})]$
$(\phi^-,\phi^+)_*(\chi^-,\chi^+)$
$*$ can be bound to any commutative and monotonic bipolar operator

Table 1: BUMP (Adapted from Zhang, 2005, 2007)

– Patients in deep depression tend to become suicidal.

We then have the question: How to characterize the neurobiological reactions of the three patients?

It is shown (Zhang, Pandurangi, and Peace, 2006, 2007) that the dynamic reactions can be characterized with a balancing operator $\oplus$, a counterintuitive oscillator $\oslash$, and a intuitive oscillator $\otimes$, respectively, defined in the bipolar lattice $B = \{-1,0\} \times \{0,1\}$, but there is no way to define such non-linear dynamic operators in the bivalent lattice $\{0,1\}$.

At this point, someone may argue that the Boolean Cartesian product $\{0,1\} \times \{0,1\}$ will solve the problem. Unfortunately, that is untrue. For instance, non-linear bipolar dynamic oscillators such as the operator $\otimes$ can be easily defined on the bipolar lattice $\{-1,0\} \times \{0,1\}$ as, $\forall (a,b),(c,d) \in B$,

$$(a,b) \otimes (c,d) = (-((|a \wedge d) \vee (b \wedge |c|)), ((|a| \wedge |c|) \vee (b \wedge d))), \tag{1}$$

which implements the bipolar oscillatory semantics $-- = +$, $-+ = -$, $+- = -$, and $++ = +$ as fundamental mathematical truth. With the Boolean Cartesian product $\{0,1\} \times \{0,1\}$ we have no way to represent the bipolar oscillation semantics for bipolar dynamic pattern analysis and classification.

Theoretically speaking, without oscillation there would be no brain dynamics, no mental balance, no equilibrium, and essentially there would be no truth-based universe. This manifests the fact that the two poles of bipolar equilibrium are indispensable in equilibrium-based bipolar cognition, knowledge representation, and logical reasoning just as the negative and positive signs "−" and "+" are indispensable in math. In turn, the doctrine of mathematical abstraction in classical set theory has to be re-examined and further extended to include bipolarity for equilibrium-based open-world holistic dynamic reasoning. We call this the bipolar crisp frontier.

The flagship of the bipolar crisp frontier is a non-linear dynamic bipolar universal modus ponens (BUMP) as shown in Table 1.

BUMP states that given equilibrium variables (φ^-,φ^+), (ψ^-,ψ^+), (ϕ^-,ϕ^+), and (χ^-,χ^+), if bipolar interaction $_*$ occurs between (φ^-,φ^+) and (ψ^-,ψ^+), the same bipolar interaction occurs between (ϕ^-,ϕ^+) and (χ^-,χ^+) provided that $[((\varphi^-,\varphi^+) \Rightarrow (\phi^-,\phi^+)) \equiv (-1,1)\ \&\ ((\psi^-,\psi^+) \Rightarrow (\chi^-,\chi^+)) \equiv (-1,1)]$.

The crisp bipolar frontier brings the ancient Chinese YinYang theory into modern sciences for reasoning with the illogical as well as logical phenomena in an open word of dynamic equilibria. According to YinYang theory, every matter consists of two sides: Yin is the negative side and Yang is the positive side. The coexistence of the two sides in equilibrium and/or harmony is considered as a key for the mental or physical health of a biological system, neurological system, multiagent system, or any dynamic system. This principle has played an essential role in the success of traditional Chinese medicine where symptoms are often diagnosed as the loss of balance and/or harmony of the two sides. Due to the lack of a formal scientific or mathematical basis, however, YinYang has largely remained mysterious for more than two thousand years, albeit surprisingly pervasive in human decision making. Indeed, YinYang has become a buzzword in every aspect of the western as well as the eastern societies including genomics research (Shi, et al, 1991; Ai, et al, 2000; Kim, et al, 2007) but still mysterious in logical reasoning. Bipolar crisp sets, therefore, opened a new frontier in mathematics.

5 Fuzzy Bipolarity and A Borderline of Isomorphism

A bipolar fuzzy set is a set of quasi- or fuzzy equilibria. That is their negative and positive poles do not have to be exactly balanced/unbalanced. The bipolar fuzzy frontier is best characterized by a borderline of isomorphism:

A Comment from Fuzzy Set Community: "All arguments stating that the bipolar lattice $[-1,0] \times [0,1]$ structurally differs from the unit square $[0,1]^2$ are dubious and/or wrong. It does not require a lot of mathematical skills to understand that the mapping $f : [-1,0] \times [0,1] \to [0,1]^2 : (x,y) \to (-x,y)$ is a bijection and, hence, the structure of $[0,1]^2$ is identical to the one of $[-1,0] \times [0,1]$ (i.e. there is a one to one correspondence)."

Discussion: In mathematics, an isomorphism is a bijective map f such that both f and its inverse f^{-1} are homomorphisms. A homomorphism is a map from one algebraic structure to another of the same type that preserves all the relevant structure; e.g. the properties such as identity elements, inverse elements, and binary operations, .., etc. In another word, isomorphism is a one-to-one correspondence between the elements of two sets such that the result of an operation on elements of one set corresponds to the result of the analogous operation on their images in the other set.

Evidently, the critic only noticed the bijective mapping between the two lattices but failed to notice that "both f and its inverse f^{-1} are homomorphisms" or maps from one algebraic structure to another of the same type that preserve all the relevant structure; i.e. properties like identity elements, inverse elements, and binary operations.

Firstly, it is easy to argue that bipolarity as a key property of bipolar equilibrium could be added to the property list to be preserved. It can be observed that the basic numbers -1 and $+1$ both present in the bipolar lattice $B = [-1,0] \times [0,1]$ such that we have $(-1) \times (-1) = 1$ and $1 \times 1 = 1$. Although the multiplicative identity element 1 is preserved in the unipolar lattice $[0,1]^2$, -1 as a multiplicative inverse of 1 was not preserved and bipolarity was lost. Therefore, we may say $B = [-1,0] \times [0,1]$ is not really isomorphic to $[0,1]^2$ from an equilibrium-based perspective unless we say -1 is similar to $+1$. But for a bipolar equilibrium, they are not. For instance, we can not label our car battery with $(+,+)$.

Secondly, we examine whether the result of an operation on elements of one set corresponds to the result of the analogous operation on their images in the other set. It can be illustrated that, without bipolarity, equilibrium-based non-linear bipolar dynamic cross-pole triangular norms[5] (oscillatory t-norms) $\otimes$ can not be meaningfully defined on $[0,1]^2$. For instance, $\forall (x,y)(u,v) \in B$,

$$(x,y) \otimes (u,v) \equiv (-((|x| \wedge v) \vee (y \wedge |u|)), ((|x| \wedge |u|) \vee (y \wedge v))) \tag{2}$$

defines a cross-pole oscillatory t-norm[5] (bipolar commutative, associative, monotonic, with identity element (0,1)) with the infused bipolar oscillation semantics $-- = +$, $-+ = -$, $+- = -$, and $++ = +$ on each pole. Without $(-,+)$ bipolarity, however, the product lattice $[0,1]^2$has no way to represent the bipolar dynamic oscillatory semantics or basic mathematical truth $-- = +$, $-+ = -$, $+- = -$, and $++ = +$. For instance, we have $(-1,0) \otimes (-1,0) = (0,1)$. But with $[0,1]^2$ the operator $\otimes$ can not be defined. Therefore, the bijective map f and its inverse f^{-1} are not really homomorphisms. Someone might say "I can define $(1,0) \otimes (1,0) = (0,1)$ for the same semantics

$--=+$." Unfortunately, we have to say that "your definition is a derivation from $B=[-1,0]\times[0,1]$ not from $[0,1]^2$. It is actually not a representation for $--=+$ but for 'left left = right'. And following your model $++=+$ will become 'right right = right' that seems to be odd. This is a typical example that shows knowledge 'flavoring' can not go beyond knowledge representation."

Structurally, while the unit square lattice $[0,1]^2$ was proposed for static truth-based reasoning with two dimensions where the words "bipolar", "dynamic", and "equilibrium" were not mentioned and bipolar dynamic interaction or oscillation was not observed, the bipolar lattice B is proposed for equilibrium-based bipolar dynamic reasoning. It is natural to use $[0,1]^2$for characterizing the truth on the x and y dimensions if they are not the two sides of one matter in opposite polarity. It would be ridiculous, however, if we use $(1,1)$ or $(+,+)$ to label the two poles of a bipolar equilibrium such as car battery, action and reaction forces, the two poles of a magnet, positive and negative electromagnetic fields, depression and mania, .., etc. Clearly, an equilibrium-based dynamic algebraic structure with two opposing poles is hardly the same type of truth-based static structure that does not have dynamic bipolar interaction.

Thirdly, according to Hofstadter's Pulitzer Prize winning book:

"The word 'isomorphism' applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where 'corresponding' means that the two parts play similar roles in their respective structures." (Hofstadter, 1979, in 1999 edition, p. 49)

Here -1 and 1 are two corresponding parts. The question to be asked then is "Does the positive number play a similar role as the negative number?" If the answer is "Yes" the two lattices would be isomorphic. If the answer is "No" the two lattices are not isomorphic. Evidently the two poles of a bipolar equilibrium can not be considered similar due to their fusion or binding in opposite polarities. That is why we use $(-1,1)$ to characterize the balance of two opposite poles and we use $(-,+)$ to label a battery. If they were similar or identical we could use $(+,+)$ to label a battery, we could say $(a+b)^2\equiv(a-b)^2$, we could allow our children to learn mathematics without the negative sign.

Historically, isomorphism was defined long before knowledge representation became a major area in automated reasoning and data mining. It is a well-known principle that without representation we can not reason on anything. Hence, without the representation of $(-,+)$ bipolarity, we can not reason on $(-,+)$ bipolarity. If the first quadrant is enough, why do we have the other three quadrants? If classical logic is enough, why do we need different fuzzy sets, t-norms, and the mathematics of uncertainty at all? Similarly, if positive numbers are enough, why do we need negative numbers and "0"? Without "0" and negative numbers can we have symmetry and bipolar equilibrium? Can we have rigorous of mathematics?

After all, isomorphism is defined as a two-way mapping for enhancing research that has never meant to be a "one-way straight positive number only" bashing stick. Since the negative lattice $[-1,0]$ is part of L-fuzzy sets as proposed in (Goguen, 1967), the bipolar lattice $[-1,0]\times[0,1]$ is logical, conceptual, computational, dynamic, and cognitive. The unipolar lattice $[0,1]^2$, on the other hand, is truth-based, static, and neither logical nor cognitive/conceptual for equilibrium-based bipolar knowledge representation because without $(-,+)$ bipolar representation we could not reason on $(-,+)$ bipolarity with the full power of mathematics.

The flagship of the bipolar fuzzy frontier is a norm-based granular version non-linear dynamic bipolar universal modus ponens (BUMP) as shown in Table 2. In the granular version, the universal operator $*$ can be instantiated to any bipolar t-norm, p-norm, or a co-norm with infinite number of gray levels for dealing with different uncertainties (Zhang, 2007).

BUMP: $(\varphi^-,\varphi^+)*(\psi^-,\psi^+)$,
$[((\varphi^-,\varphi^+) \Rightarrow (\phi^-,\phi^+)) \equiv (-\mathbf{1},\mathbf{1})$ & $((\psi^-,\psi^+) \Rightarrow (\chi^-,\chi^+)) \equiv (-\mathbf{1},\mathbf{1})]$
$(\phi^-,\phi^+)*(\chi^-,\chi^+)$
($*$ can be bound to any commutative and monotonic bipolar fuzzy norm)

Table 2: Norm-Based Granular BUMP (Adapted from Zhang, 2007)

6 Conclusions

While some progress has been reported, many challenges lay ahead. It is fair to say that the fuzzy community has made great progresses on the fuzzy set and logic theory frontier with a successful position in its challenge to the doctrine of classical set theory. We have so far witnessed only very limited success in the YinYang bipolarity double frontier. As the matter of fact, the frontier has just been opened and not been fully explored yet.

Despite the unreasonable bashing on equilibrium-based bipolarity (due to misunderstanding or conflict research interests or both), the authors of this paper are encouraged to report that Professor Lotfi A. Zadeh recently brought forward an inspiration to the future of the bipolarity double frontier byrecognizing "bipolar fuzzy sets" on Scholarpedia (Zadeh, 2007). It is hopeful that such inspiration will chill some of the unreasonable bashing and lead to healthier further discussions on related topics. It is also hopeful that the non-linear bipolar universal modus pones (BUMP) will open the "classical" gate of the "closed world" and relinquish the power of equilibrium-based open-world dynamic reasoning.

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A Review Of Fuzzy System Models: Fuzzy Rulebases vs Fuzzy Functions

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Abstract. Fuzzy System Modeling (FMS) is one of the most prominent tools that can be used to identify the behavior of highly nonlinear systems with uncertainty. In the main, FMS techniques utilized Type 1 fuzzy sets in order to capture the uncertainty in the system. We review Type 1 fuzzy system models known as Zadeh, Takagi-Sugeno, Türkşen and Celikyilmaz-Türkşen models. Zadeh and Takagi- Sugeno models are essentially, Fuzzy Rule Base, FRB, models where as Türkşen and Celikyilmaz-Türkşen models are essentially Fuzzy Function, FF, models.
In data-driven fuzzy system modeling methods discussed here, a fuzzy clustering algorithm is used in order to identify the fuzzy system structure, i.e., either the number of fuzzy rules or alternately the number of fuzzy functions.

Index terms — fuzzy system models, fuzzy functions, fuzzy clustering, and Type 1 fuzzy system models.

1 Introduction

More than 40 years of research have demonstrated that fuzzy system models are the most successful models to handle uncertainties in decision-making. The major advantages of fuzzy system models are their robustness and transparency. Fuzzy system modeling achieves robustness by using fuzzy sets which are represented by membership values that incorporate imprecision in system models. In addition, unlike some other system models, such as neural networks, the fuzzy system models are highly descriptive and somewhat transparent with cluster representations.

In the last two decades, researchers proposed several data driven Type 1 fuzzy system modeling approaches that can extract the hidden rules of a system behavior automatically by using historical data. The system modeling methods, proposed by Celikyılmaz-Türkşen [3,4,6], Emami et. al. [8] , Nakanishi et. al. [10] , Sugeno and Yasukawa [11] , Takagi-Sugeno [12] , and Türkşen [15] are among the most notable ones. Since these methods utilize only the historical data, i.e., they do not require expert knowledge, they are strtictly data-driven modeling techniques. Thus, in addition to being robust and transparent, these system-modeling techniques can identify system model structure objectively for a given performance measure.

In these fuzzy system models, the structure is characterized by Type 1 fuzzy sets defined by their membership values. Type 1 fuzzy sets and their membership values, defined on a universe of discourse, map an element of a system onto a precise number in the unit interval $[0,1]$, i.e., a membership value. In this paper, we review only the Type 1 fuzzy system models as the historically significant and successful modeling activity of the past in the domain of fuzzy control system and fuzzy decision support problems. We leave a review the Type 2 fuzzy system models as the potential future modeling activity in the domain of essentially fuzzy decision support system problems for another paper.

In an historical sense, Zadeh [18] , and Takagi-Sugeno [12] versions of Type 1 fuzzy system models are typically basic "Fuzzy Rule Base", FRB, models. In contrast, Type 1 "Fuzzy Function", FF, models, recently proposed by Türkşen [5] and further developed by Celikyilmaz-Türkşen [3,4,6] are alternate functional representations of fuzzy rule base models. The FF's produce better predictions in comparison to type 1 fuzzy rule base models [3,4,6,15]. Furthermore, fuzzy rule base models, in general, can not capture the interactive nature of a problem space due to projection anomalies. Whereas Fuzzy Function models are able to capture the interactions of all variables since they are not subject to projection anomalies.

The rest of this paper is organized as follows: the basic notation, terminology as well as the three well-known Type 1 fuzzy system model together with their essential structures i.e., Zadeh, Takagi-Sugeno, and Türkşen and Celikyilmaz-Türkşen prototypes, will be briefly reviewed in section II with an emphasis on the more recently proposed "Fuzzy Functions" of Türkşen [15]. An experimental comparison between "Fuzzy Rule Based" models and recently proposed "Fuzzy Function" models will be presented in section III. Finally, the conclusions will be drawn and the future research directions will be provided in section IV.

2 Type 1 fuzzy system models

In general fuzzy system models identify an underlying relationship between input and output variables of a system. In this paper, we deal only with Multi-Input Single Output (MISO) Type 1 fuzzy system models. Generally fuzzy system models represent relationships between the input and output variables as a collection of: (a) either if-then rules that utilize linguistic labels, i.e., fuzzy sets or (b) fuzzy functions that takes membership values as well as selected input variables and their transformations as their arguments.

2.1 Type 1 Fuzzy Rule Bases

The general If-Then fuzzy rule base structure can be written as follows:

$$R : \underset{i=1}{\overset{c*}{\text{ALSO}}} (\text{IF } \mathit{antecedent}_i \text{ THEN } \mathit{consequent}_i) \ , \tag{1}$$

where c^* is the number of rules in the rulebase. There are several well known fuzzy rulebase structures which mainly differ in the representation of their consequents. If the consequent is represented with fuzzy sets then the rulebase can be categorized as the Zadeh Fuzzy Rulebase, Z-FRB [18] A particular applied version of this is known as Sugeno-Yasukawa [11] , Fuzzy Rule Based, SY-FRB. Whereas, if the consequent is represented with a linear equation of input variables, then the rulebase structure is known as Takagi-Sugeno Fuzzy Rulebase (TS-FRB) structure [12] . The Z-FRB alterably SY-FRB and TS-FRB structures can be formalized as follows:

Let nv be the number of input variables in the system. Then, the multidimensional antecedent, X, can be defined as $X = (x_1, x_2, \ldots, x_{nv})$, where x_j is the jth input variable of the antecedent in the domain of X. X can be defined as $X = X_1 \times X_{21} \times \ldots \times X_{nv}$, where $X_j \subseteq \mathfrak{R}$ is the domain of variable x_j. Similarly, the domain of the output variable, , will be denoted as $Y \subseteq \mathfrak{R}$. Then, the ith rule, R_i, and rulebase, R, in Z-FRB(SY-FRB) structure can be defined as:

$$R_i : \text{IF} \underset{j=1}{\overset{NV}{AND}} (x_j \in X_j \text{ isr } A_{ji}) \text{ THEN y} \in Y \text{ isr } B_i, \forall i = 1, .., c^* \tag{2}$$

$$R : \underset{i=1}{\overset{c^*}{ALSO}} \left(\text{IF } \underset{j=1}{\overset{NV}{AND}} (x_j \in X_j \text{ isr } A_{ji}) \text{ THEN y} \in Y \text{ isr } B_i \right) \ , \tag{3}$$

where A_{ji} is the linguistic label associated with j^{th} input variable of the antecedent in the i^{th} Rule, r_i, with membership function $\mu_i(x_j) : x_j \to [0,1]$ and similarly B_i is the consequent linguistic label of the i^{th} rule with membership function $\mu i(y) : y \to [0,1]$, and c^* is the number of rules in the model. The above structure assumes no interactivity between input variables because the membership functions of every A_{ji} is obtained by the projection of $nv + 1$ dimensional internal system representation. It should be noted that Sugeno-Yasukawa [11], SY-FRB, is a surrogate version of Z-FRB in applied studies. In order to eliminate the no interactivity assumption, Delgado, et. al. [7], Babuska, et. al. [1], and Uncu and Türkşen [16] used multidimensional Type 1 fuzzy sets to represent the antecedent part of the rules. Hence, the Z-FRB (SY-FRB) structure can be expressed as follows:

$$R : \underset{i=1}{\overset{c*}{ALSO}} (\text{IF } x \in X \text{ isr } A_i \text{ THEN y} \in Y \text{ isr } B_i) \ , \tag{4}$$

where the multidimensional antecedent fuzzy set of i^{th} rule is defined as $\mu_i(x) : x \to [0,1]$. The other well-known fuzzy rulebase structures, namely Takagi-Sugeno (TS-FRB) fuzzy rulebase structure, can be expressed, respectively, as follows:

$$\underset{i=1}{\overset{c^*}{\text{ALSO}}} \left(\text{IF } x \in X \text{ isr } A_i \ \text{ THEN } \ y_i = a_i x^T + b_i \right) \ , \tag{5}$$

where, $a_i = (a_{i,1}, \ldots, a_{i,nv})$ is the regression coefficient vector associated with the i^{th} rule, b_i is the scalar associated with the i^{th} rule.

2.2 Type 1 Fuzzy Functions

There are at least two ways to form fuzzy functions: (i) Fuzzy Functions estimated by Least Squares, FF-LSE , of Türkşen [15] and (ii) Fuzzy Functions estimated by Support Vector Machines, FF-SVM 's, of Çelikyilmaz and Türkşen [4] . They are structurally different from Zadeh's [18] linguistic fuzzy rule bases (Z-FRB), (SY-FRB) or Takagi-Sugeno fuzzy rulebase TS-FRB [12] and "Fuzzy

Regression" models of Tanaka, et. al. [13,14] and its variations, and Hathaway and Bezdek [9] model. In particular, the proposed "Fuzzy Functions", FF approach introduces membership values and their transformations as new input variables in addition to the original scalar input variables in fuzzy function estimations. In FF approach one first executes a fuzzy clustering algorithm with original selected input variables after an execution of a feature selection algorithm; and then determines (local) optimum number of fuzzy clusters and hence the associated membership values. Fuzzy membership values, obtained in these two previous steps then become (a) new input variable (s) with their potential transformations. Finally a fuzzy function to represent each fuzzy cluster separately can be identified. Thus there are as many fuzzy functions as there are fuzzy clusters similar to Hathaway and Bezdek's [9] Fuzzy C-Regression model (FCRM). But Hathaway and Bezdek use membership values as the weights to be used in the estimation of the functions using weighted least squares algorithm. FCRM updates the membership values as the similarity measure by using estimation error from these functions.

Where as "Fuzzy Functions"of Türkşen [15] and Çelikyılmaz and Türkşen [3] are estimated with original input variables as well as new input variables generated with membership values obtained for each cluster from FCM algorithm. Therefore it is structurally a new and unique approach for the determination of fuzzy system models in place of fuzzy rule bases. "Fuzzy Functions", FF, represent fuzzy rule bases indirectly. When the relationship between input variables and the output variable of the system can be explained in the in the newly generated interactive multi-dimensional space of the data, it is quite reasonable and faster to estimate the "Fuzzy Functions" using least squares estimation. When this relationship is more complex and there needs to be a non-linear transformation of the original input variables, it is better to map the input dataset into a higher dimensional space, e.g., a hyper-space, where the input dimension is large (maximum n). One of the powerful methods to find the "Fuzzy Functions" which defines a linear relation between input and output variables in the higher dimension, but a non-linear relationship in the original dimension is the Support Vector Machines, SVM, which was first proposed by Vapnik [17]. For the regression studies, a regression algorithm can be applied to find the fuzzy functions. Hence, in the next sections, we are going to specify the details of the fuzzy functions estimated using the Least Squares Estimates, LSE, method. We have also implemented Support Vector Machine for Regression, SVR, algorithms in Çelikyılmaz and Türkşen [4]

It is to be noted for the sake of emphasis that the estimated parameters of the inputs are not fuzzy sets in our proposed approach in contrast to Tanaka, et.al. [13,14] approaches. It should be recalled that in FF's membership values and their transformations are augmented into the original selected input set as new and additional variables. In our experience, it is found that FF approach is most suitable for those analysts who are familiar with a function estimation technology, e.g., the least squares technology, support vector machines, ridge regression, etc. They only need to develop an understanding of some fuzzy clustering algorithm without studying many aspects of fuzzy theory. All they have to understand is the notion of membership values and how they can be obtained from a fuzzy clustering algorithm such as FCM [2] and/or its variations such as IFCM [5] in addition to their usual background knowledge of a function estimation technique, e.g., LSE, or SVR, etc.

Thus FF method is a novel approach in order to provide an easy entry into fuzzy system modeling for mathematicians and statisticians who are working in industry and for other novices. For this purpose, we present next our generalization of the LSE algorithm, which includes membership values and their transformations in addition to the original scalar input variables.

2.3 Fuzzy Functions with LSE(FF-LSE)

In the Ordinary LSE (OLSE) method, the dependent variable, y, is assumed to be a linear function of one or more independent, input, variables, x, plus an error component as follows:

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_{nv} x_{nv} + \varepsilon \ ,$$

where y is the dependent output, x_j's are the inputs (explanatory variables), for $j = 1,\ldots,nv$, nv is the number of selected inputs and ε is the independent error term which is typically assumed to be normally distributed. The goal of the least squares method is to obtain estimates of the unknown parameters, β_j's, $j = 0,1,\ldots,\ nv$, which indicate how a change in one of the independent variables affects the dependent variable as follows:

$$\beta = (X'X)^{-1}X'Y \ , \tag{6}$$

where $\beta = (\beta_0,\beta_1,\ldots,\beta_{nv})$. The proposed generalization of OLSE as FF-LSE, requires that a fuzzy clustering algorithm, such as FCM [2] or IFCM [5] , be available to determine the interactive (joint) membership values of input-output variables in each of the fuzzy clusters that can be identified for a given training data set.

Let (x_k,y_k), $k = 1,\ldots,nd$, be the set of observations in a training data set, such that

$$X = (x_{j,k} \mid j = 1,\ldots,nv, k = 1,\ldots,nd) \ . \tag{7}$$

First, one determines the optimal (m^*,c^*) pair for a particular performance measure, i.e., a cluster validity index, with an iterative search and an application of FCM algorithm, where m is the level of fuzziness (in our experiments, we usually take $m = 1.1,\ldots,2.5$), and c is the number of clusters (in our experiments, we usually take $c = 2,\ldots,10$). The well known FCM optimization can be stated as follows:

$$\begin{aligned} \min J(U,V) &= \sum_{k=1}^{nd}\sum_{i=1}^{c}(u_{ik})^m(\|x_k - v_i\|)_A \\ s.t.\ 0 &\le u_{ik} \le 1, \forall i,k \\ \sum_{i=1}^{c} u_{ik} &= 1, \forall k \\ 0 &\le \sum_{k=1}^{nd} u_{ik} \le nd, \forall i \ , \end{aligned} \tag{8}$$

where J is objective function to be minimized,$\|\cdot\|_A$ is a norm is a norm that specifies a distance-based similarity between the data vector x_k and a fuzzy cluster center v_i. In particular, $A = I$ is the Euclidian norm and $A = COV^{-1}$ is the Mahalonobis norm, etc., where COV is the covariance matrix.

The optimal pair, (m^*,c^*), can be determined with a user defined cluster validity index, partition entropy or partition coefficient [2] . Another alternative of selecting the optimum pair would be running the overall FF-LSE model for every (m,c)pair specified by the user and determining the optimal pair from the training RMSE values of each model [3,4,6,15]. The experiments in this paper follow the second alternative. The determination of the optimal (m^*,c^*) requires an implementation of the iterative clustering algorithm [2,5]

Once the optimal pair (m^*, c^*) is determined with the application of FCM algorithm, one next identifies the cluster centers for $m = m*$ and each cluster $i = 1, \ldots, c*$ as:

$$\underset{m*}{v_{X|Y,i}} = (x_{1,i}, x_{2,i}, \cdots, x_{nv,i}, y_i) \tag{9}$$

From this, we identify the cluster centers of the "input space" again for $m = m^*$ and $c = 1, \ldots, c^*$ as:

$$\underset{m*}{v_{X,i}} = (x_{1,i}, x_{2,i}, \cdots, x_{nv,i}) \tag{10}$$

Next, one computes the normalized membership values of each data sample in the training data set with the use of the cluster center values determined in the previous step. There are generally two steps in these calculations:

(a) first we determine the (local) optimum membership values u_{ik}'s and then determine μ_{ik}'s that are above an α-cut in order to eliminate harmonics generated by a clustering algorithm:

$$u_{ik} = \left(\sum_{j=1}^{c} \left(\frac{\|x_k - v_{X,i}\|}{\|x_k - v_{X,j}\|} \right)^{\frac{2}{m-1}} \right)^{-1}, \ \mu_{ik} = \{u_{ik} \geq \alpha\}, \tag{11}$$

Where μ_{ik} denotes the membership value of the kth vector, $k = 1, \ldots, nd$, in the i^{th} rule, $i = 1, \ldots, c^*$ and x_k denotes the k^{th} vector.

(b) next, we normalize them as:

$$\gamma_{ik}(x_k) = \frac{\mu_{ik}(x_k)}{\sum_{i'=1}^{c} \mu_{i'k}(x_k)} \ , \tag{12}$$

where $\gamma_{ik}(x_k)$'s are the normalized membership values of a data sample x_k in the ith cluster, $i = 1, \ldots, c^*$, which in turn indicate the membership values that will constitute as a new input variable in our proposed scheme of function identification for the representation of ith cluster. Let $\Gamma_i = (\gamma_{ik} \mid i = 1, ..., c^*; k = 1, ..., nd)$ be the membership values of a data x_k sample in the ith cluster, i.e., ith rule.

Next we determine a new augmented input matrix of X_i for each $i = 1, \ldots, c^*$, of the clusters, which could take on several forms depending on which transformation of membership values we want to or need to include in our system structure identification for our intended system analyses. Examples of possible augmented input matrices which were investigated in [15] are:

$$X_i' = [1, \Gamma_i, X], \text{ or,} \tag{13}$$
$$X_i'' = \left[1, \Gamma_i^2, X\right], \text{ or,} \tag{14}$$
$$X_i''' = \left[1, \Gamma_i^2, \Gamma_i^m, \exp(\Gamma_i), X\right], \text{ etc.,} \tag{15}$$

where X_i', X_i'', X_i''' are potential augmented input matrices to be used in a least squares estimation of a new system structure identification and $\Gamma_i = (\gamma_{i,k} \mid i = 1, ..., c^*; k = 1, ..., nd)$. The choice amongst X_i', X_i'', X_i''', etc., depends on whether we want to or need to include just the membership values or some of their transformations as new input variables in order to obtain the best representation of a system behavior. A new augmented input matrix having a single input variable in the original input space when only membership values themselves are augmented to the dataset, i.e., X_i' may look like as shown below

$$X_i' = [1, \Gamma_i, X] = \begin{bmatrix} 1 & \gamma_{i,1} & x_{i,1} \\ \vdots & \vdots & \vdots \\ 1 & \gamma_{i,nv} & \gamma_{i,nv} \end{bmatrix}$$

Up to this point, in the proposed system modeling approach, we have defined how the augmented input matrix for each cluster could be formed using FCM algorithm. Both the proposed FF-LSE and FF-SVM approaches implement these steps. From this point forward, the estimation of fuzzy functions takes place for each cluster, where one can implement any function estimation methodology, e.g., LSE or SMV. Different approaches are followed in the estimation of fuzzy functions using the augmented matrices. Here we continue to specify the FF-LSE models as an example.

Thus the function of a single input-single output model, which includes in addition only the membership values as the additional input variable, $Y_1 = \beta_{i0} + \beta_{i1}\Gamma + \beta_{i2}X$, that represents the i^{th} rule corresponding to the i^{th} interactive (joint) cluster in space, would be estimated with FF-LSE approach as follows:

$$\beta_i^* = (X_i'^T X_i')^{-1}(X_i'^T Y_i) \ , \tag{16}$$

where $\beta_i^* = (\beta_{i0}^*, \beta_{i1}^*, \beta_{i2}^*)$ and $X_i' = [1, \Gamma_i, X]$, provided the inverse of covariance matrix, $(X_i'^T X_i')^{-1}$, exists. The estimate of y

$$Y_i^* = \beta_{i0}^* + \beta_{i1}^* \Gamma_i + \beta_{i2}^* X \tag{17}$$

The single output value, Y, is calculated using each output value, one from each cluster, and weighting them with their corresponding membership values as follows:

$$Y^* = \frac{\sum_{i=1}^{c*} \gamma_i Y_i^*}{\sum_{i=1}^{c*} \gamma_i} \tag{18}$$

Within the proposed framework, the general form of the shape of a cluster for the case of a single input variable X_j and for the ith cluster can be conceptually captured by a second order (idealized cone) function when one introduces the square of membership values into the augmented input matrix in the space of $[U \times X \times Y]$ which can be illustrated with a prototype shown in Figure 1 .

In a number of real life case studies, we have in fact found out that generally some second order or exponential functions give a good approximation from amongst some 20 alternatives we have experimented.

3 CASE STUDY APPLICATIONS

In order to test the proposed Fuzzy Function model performances as opposed to Fuzzy Rule Base systems three input-output datasets are considered in this investigation. These are:

i. Daily price of a stock in stock market.
ii. Customer Income Prediction model for a major bank.
iii. The amount of chemicals for a desulphurization process for a steel processing company.

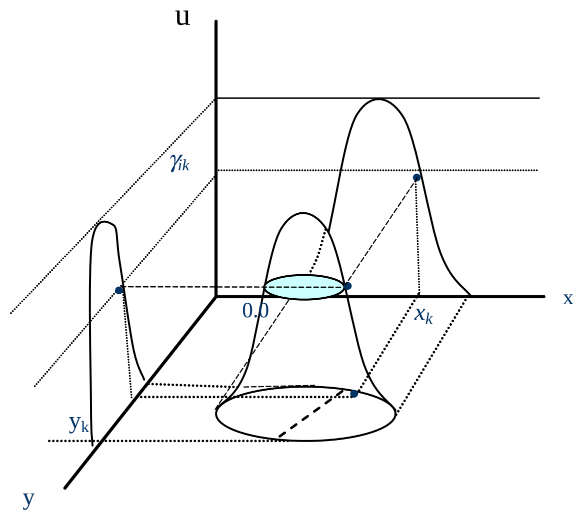

Fig. 1. A fuzzy cluster in $[U \times X \times Y]$ space.

The specifications of each datasets are discussed in the following parts

Daily Stock Price Dataset:

Daily stock price dataset comprises of the daily trend data of stock prices. This dataset was introduced by Sugeno and Yasukawa [11] This same data set which was also used in Nakanishi, et.al, [10] as a case study has been used in various other studies.

Out of 100 observations, 50 of them are used for the training purposes and the other 50 was hold-out for testing purposes. There were originally 11 input variables and a single output variable in the dataset. Based one Preliminary input selection method only 4 of input variables, i.e., x_2, x_4, x_8, x_{10}, were found to have importance on the output variable. The rest of the variables had insignificant effect on the output.

Income Prediction Dataset:

The purpose of the Income Prediction Model was to predict the income of the future customers based on the current customer information and 1996 year census data. There were more than 200 variables and hundreds of thousands of customers. According to business needs, the dataset was partitioned into 9 different parts based on age, number of different types of investment accounts held by customers and different regions of residency. In this study, only one partition was investigated.

We have only selected 10% of the i.i.d. data samples to do our research on using only single partition explained above. The data was cleaned form the outliers using the expert's knowledge and 900 training and 900 testing observations are selected randomly. The dataset was comprised of 11 input variables based on the correlation analysis, 3 input variables were discarded from the dataset resulting in 8 input variables. There were no census variables among the selected input variables.

Desulphurization Dataset:

A torpedo car desulphurization facility removes sulfur from hot metal leaving the blast furnaces before it is sent to the next process. Generally, desulphurization is carried out by injecting two different powdered reagents directly into the hot metal via a lance. The reagents react with the sulfur in the hot metal and residue, which is rich in sulfur, is separated from the iron.

The aim of the fuzzy data-mining and fuzzy system modeling project was to determine the right amounts of the reagents to be added into the hot metal. These regents are expensive materials and precise estimation is required. There are vast quantities of data available on the desulphurization process, which has various characteristics. There are 750 training and 900 verification samples used in the experiment. Based a preliminary input selection method only 5 input variables are found to be important.

A. Experimental Design

Using the three different input-output datasets, we build three different fuzzy system model structures, FF-LSE, SY-FRB and TS-FRB. In order to keep the consistency between each model structure, the same training and testing datasets are used for the four fuzzy system models with the same input variables. The categorical variables are transformed into probabilities using logistic regression and are used as additional inputs only in Income Prediction and Desulphurization Datasets in all of the four models.

B. Sugeno-Yasukawa Models and Takagi-Sugeno Models

The proposed FSM models are compared to two well known Fuzzy Rule Base Models: (i) Sugeno and Yasukawa [11] fuzzy logic based approach using Partition Type Fuzzy Model, SY-FRB, as a surrogate to Z-FRB (ii) Takagi and Sugeno's [12] fuzzy system modeling approach, TS-FRB. In Sugeno and Yasukawa's FSM approach, they use linguistic measures for both the consequent and the antecedent part of the fuzzy rules where the system learns all its inference parameters from the data without an expert's intervention. The variable selection method defined in their paper is not applied in this study to these 3 datasets in order to compare the models on the same basis.

In Takagi and Sugeno's FRB (TS-FRB) structure [12] , they assume that the antecedent membership functions are to be characterized with triangular membership functions. In their approach, each input variable space is assumed to be partitioned into two clusters and logical connective AND is taken as MIN. Then, the structure identification problem is just to identify the regression equation coefficients for each rule and the antecedent parameters for each input variable in each rule. Researchers proposed several structure identification methods to identify the membership functions from the data, e.g., Delgado et.al.,[7] Babuska et. al., [1] , etc. In this paper we have used Babuska et. al.'s [1] modified Takagi-Sugeno study where the membership functions of the antecedents are identified using Fuzzy C-Means clustering and projected onto each input vector. The degree of fulfillment of each rule is then calculated using a t-norm operator, i.e., product operator. Only one aggregate input membership function is identified for each rule. The inference parameters are the same as the traditional Takagi-Sugeno inference method [12] where the relationship with the consequents and the antecedents are assumed to come from a linear function.

C. Fuzzy Functions Models

In this paper, the modeling performance from the Fuzzy Functions with LSE models, FF-LSE are compare to the Fuzzy Rule Base structures. Instead of using cluster validity indices to select the optimum model parameters, we measured the optimum model based on the best model performance using RMSE by applying a grid search for each parameter. Note that Fuzzy Functions with LSE system models have 2 parameters, which are the FCM parameters, i.e., degree of fuzziness and cluster size.

In these fuzzy function models, membership values and their exponential transformations are used as additional input variables. For the fuzzy functions with LSE models 2 parameters are specified for each model. The model performance of each model is determined using Root Mean Square Error, RMSE, of the models as follows:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{N}(y_i - \hat{y}_i)^2} \tag{19}$$

y_i, and $\hat{y}_i$ are the actual and estimated output values of a single observation, N is the total number of observations in the dataset.

D. Model Results

The three fuzzy system models are determined for: (i) the daily stock price of a stock market, (ii) income prediction, and (iii) reagents estimation in desulphurization process datasets. The results are displayed in Table 1, 2 and 3.

"Fuzzy Function", FF, models, when estimated with an LSE algorithm, show better generalization than the Fuzzy Rule Base FRB models. The "Fuzzy Function", FF, models can increase the model performance from 1.2% up to 43% depending on the dataset in test data cases.

	FF-LSE	TS-FRB	SY-FRB
RMSE(train)	3.82	2.76	7.16
RMSE(test)	5.61	5.68	9.93

Table 1: Daily Stock Price of a Stock in Stock Market

	FF-LSE	TS-FRB	SY-FRB
RMSE(train)	0.52	0.49	0.58
RMSE(test)	0.64	0.80	0.70

Table 2: Income Prediction Dataset*

* The RMSE values are calculated from standardized output values.

	FF-LSE	TS-FRB	SY-FRB
Reagent1			
RMSE(train)	40	35	69.5
RMSE(test)	45	45	72
Reagent2			
RMSE(train)	6.49	5.62	10.01
RMSE(test)	7.19	7.19	10.80

Table 3: Reagent estimation for Desulphurization Processt

Table 4. displays the optimum parameters of the models from each experiment whose results are displayed in Table 1-3. In Table 4, *m* refers to the degree of fuzziness (weighting exponent) of the Fuzzy C-Mean clustering algorithm, *c* indicates the number of clusters models.

Dataset	Model Type	Optimum model Parameters
Daily Stock Price	FF-LSE	$c = 8$, $m = 1.6$
Income Prediction	FF-LSE	$c = 8$, $m = 1.6$
Desulphurization Process	FF-LSE	Reagent 1: $c = 5$, $m = 1.4$
		Reagent 2: $c = 6$, $m = 1.5$

Table 4: Optimum Model Parameters of three datasets.

The grid search algorithms applied in this paper try to find the best RMSE value from training data in each experiment and assign these parameters as the optimum model parameters. The algorithm searches for the minimum regression error. Then, verification dataset output is inferred using the optimum parameters. The issue with these grid search algorithms is that, sometimes, the models get stuck in the local minimum which is smaller than the global minimum and this might cause generalization problems. An example to this concept is shown in income prediction dataset (Table 2.). The model parameters best fit to the training data when FF-SVM is used but this causes generalization problems. It should also be reminded that, when there is a linear relationship between the inputs and the output, then LSE model performances will be as good as the other model performances. On the other hand, FF-LSE models, in three of the datasets, show more reliable results than the SVM models. One should run both models and determine the optimum model parameters after observing the results from both models.

4 Conclusions and future study

In this paper, we have outlined basic well known three Type 1 system models, namely, Z-FRB (SY-FRB), SY-FRB, Türkşen's FF's. In particular, we have reviewed in detail: (1) Type 1 Fuzzy Rule bases and (2) Type 1 Fuzzy Functions. As well, we have demonstrated that "Type 1 Fuzzy Functions" provide better results then "Type 1 Fuzzy Rule Base" models in general in three specific case studies.

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